ON FIXED POINTS OF TWO ASYMPTOTICALLY QUASI NONEXPANSIVE MAPPINGS

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ABSTRACT
In this paper, we study the weak and strong convergence of an iterative scheme of the Ishikawa type for two strongly asymptotically quasi nonexpansive mappings in a uniformly convex Banach space. To prove our results, we modify the method of proofs, used by Fukhar-ud-din and Khan [Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces, Comp. Math. Appl. 53 (9) (2007), 1349-1360].

Keywords: Iterative scheme with errors, Condition (AR), Asymptotically nonexpansive mappings, Strongly asymptotically nonexpansive mappings, Common fixed point, Convergence.

1 INTRODUCTION
Throughout this paper we assume that $E$ is a real Banach space, $C$ a closed convex subset of $C$ and $T$ a self-mapping of $C$. We denote by $F(T) := \{x : Tx = x\}$ a set of fixed points of $T$ and by $D(T)$ a domain of $T$. We also assume that $F(T) \neq \emptyset$.

Definition 1 (Opial 1967)). $E$ is said to satisfy Opial condition, if for each sequence $\{x_n\}$ in $E$, the condition that the sequence $x_n \to x$ weakly implies that
\[
\lim_{n \to \infty} \sup \|x_n - x\| < \lim_{n \to \infty} \sup \|x_n - y\|
\]
for all $y \in E$ with $y \neq x$.

Definition 2 (Opial 1967)). Let $C$ be a nonempty subset of a normed space $E$. A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in $C$ and each $x \in E$, $x_n \to x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Definition 3. $T$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in $C$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x^* \in C$.

Definition 4. $T$ is said to be nonexpansive if for all $x, y \in C$, the following inequality holds:

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$ 

Definition 5 (Goebel and Kirk 1972). $T$ is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C, \ n \geq 1.$$ 

Definition 6. $T$ is said to be strongly asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C, \ n \geq 1.$$ 

Definition 7. $T$ is said to be strongly asymptotically quasi-nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ such that

$$\|T^n x - x^*\| \leq k_n \|x - x^*\|, \text{ for all } x \in C, x^* \in F(T) \text{ and } n \geq 1.$$ 

Definition 8. $T$ is said to be uniformly $L$–Lipschitzian if for some constant $L > 0$,

$$\|T^n x - T^n y\| \leq L \|x - y\| \text{ for all } x, y \in C, \ n \geq 1.$$ 

In recent years, Mann and Ishikawa iterative schemes (Ishikawa 1974), (Mann 1953) have been studied extensively by many authors.

For a nonempty convex subset $C$ of a normed space $E$ and $T : C \to C$,

(a) The Mann iteration process is defined by the following sequence $\{x_n\}$:

$$\begin{cases}
  & x_1 \in C, \\
  & x_{n+1} = (1 - b_n) x_n + b_n Tx_n, \ n \geq 1,
\end{cases}\tag{1}$$

where $\{b_n\}$ is a sequence in $[0, 1]$.

(b) The sequence $\{x_n\}$, defined by

$$\begin{cases}
  & x_1 \in C, \\
  & x_{n+1} = (1 - b_n) x_n + b_n Ty_n, \ n \geq 1,
\end{cases}\tag{2}$$

where $\{b_n\}$, $\{b'_n\}$ are sequences in $[0, 1]$, is known as the Ishikawa (Ishikawa 1974) iteration process.

In 1995, Liu (Liu 1995) introduced iterative schemes with errors as follows:

(c) The sequence $\{x_n\}$ in $C$ iteratively defined by:

$$\begin{cases}
  & x_1 \in C, \\
  & x_{n+1} = (1 - b_n) x_n + b_n Ty_n + u_n, \\
  & y_n = (1 - b'_n) x_n + b'_n Tx_n + v_n, \ n \geq 1
\end{cases}\tag{3}$$

where \( \{b_n\}, \{b'_n\} \) are sequences in \([0, 1]\) and \( \{u_n\}, \{v_n\} \) are sequences in \( C \); satisfying \( \sum_{n=1}^{\infty} \|u_n\| < \infty, \sum_{n=1}^{\infty} \|v_n\| < \infty \), is known as Ishikawa iterative scheme with errors.

(d) The sequence \( \{x_n\} \) iteratively defined by:

\[
\begin{align*}
    x_1 & \in C, \\
    x_{n+1} &= (1 - b_n)x_n + b_nTx_n + u_n, \quad n \geq 1
\end{align*}
\]

where \( \{b_n\} \) is a sequence in \([0, 1]\) and \( \{u_n\} \) a sequence in \( C \) satisfying \( \sum_{n=1}^{\infty} \|u_n\| < \infty \), is known as Mann iterative scheme with errors.

While it is clear that consideration of error terms in iterative schemes is an important part of the theory, it is also clear that the iterative schemes with errors introduced by Liu (Liu 1995), as in (c) and (d) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (c) and (d) which say that they tend to zero as \( n \) tends to infinity are, therefore, unreasonable.

In 1998, Xu (Xu 1998) introduced a more satisfactory error term in the following iterative schemes:

(e) The sequence \( \{x_n\} \) iteratively defined by:

\[
\begin{align*}
    x_1 & \in C, \\
    x_{n+1} &= a_nx_n + b_nTx_n + c_nu_n, \quad n \geq 1
\end{align*}
\]

with \( \{u_n\} \) a bounded sequence in \( C \) and \( a_n + b_n + c_n = 1 \), is known as Mann iterative scheme with errors.

(f) The sequence \( \{x_n\} \) iteratively defined by:

\[
\begin{align*}
    x_1 & \in C, \\
    x_{n+1} &= a_nx_n + b_nTy_n + c_nu_n, \\
    y_n &= a'_nx_n + b'_nTx_n + c'nv_n, \quad n \geq 1
\end{align*}
\]

with \( \{u_n\}, \{v_n\} \) bounded sequences in \( C \) and \( a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n \), is known as Ishikawa iterative scheme with errors.

Note that the error terms are now improved. Mann and Ishikawa iterative schemes follow as special cases of the above schemes respectively.

Many authors (Agarwal et al. 2002), (Das and Debata 1986), (Schu 1991), (Shahzad and Udomene 2006) have studied the two mappings case of iterative schemes for different types of mappings. Note that two mappings case has a direct connection with the minimization problem, see for instance (Takahashi 2000).

Let \( C \) be a nonempty convex subset of a normed space \( E \) and \( S, T : C \to C \) be two mappings.

Recently, Agarwal et al. (Agarwal et al. 2002) studied the iteration process for a couple of quasi-contractive mappings.

(g) The sequence \( \{x_n\} \) iteratively defined by:

\[
\begin{align*}
    x_1 & \in C, \\
    x_{n+1} &= a_nx_n + b_nSy_n + c_nu_n, \\
    y_n &= a'_nx_n + b'_nTx_n + c'nv_n, \quad n \geq 1
\end{align*}
\]

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0, 1]\) such that \(a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n\) and \(\{a_n\}, \{b_n\}, \{c_n\}\) are bounded sequences in \(C\).

Shahzad and Udomene (Shahzad and Udomene 2006) introduced the following iterative scheme with errors for two asymptotically quasi-nonexpansive mappings.

(h) The sequence \(\{x_n\}\) is defined by

\[
\begin{align*}
x_{n+1} &= a_n x_n + b_n S^n y_n + c_n u_n, \\
y_n &= a_n x_n + b_n T^n x_n + c_n v_n, \quad n \geq 1,
\end{align*}
\]

(S1)

where \(\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}\) are sequences in \([0, 1]\) such that \(a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n\) and \(\{u_n\}, \{v_n\}\) are bounded sequences in \(C\).

(i) It can be easily seen that for \(c_n = 0 = c'_n\), (S1) reduces to

\[
\begin{align*}
x_{n+1} &= (1 - b_n) x_n + b_n S^n y_n, \\
y_n &= (1 - b'_n) x_n + b'_n T^n x_n, \quad n \geq 1,
\end{align*}
\]

(S2)

where \(\{b_n\}, \{b'_n\}\) are sequences in \([0, 1]\).

Similar iterative schemes for one mapping can be obtained from (S1) and (S2).

(j) From (S2) we recover the following modified Mann type iterative scheme (Mann 1953),

\[
\begin{align*}
x_{n+1} &= (1 - b_n) x_n + b_n T^n x_n, \quad n \geq 1,
\end{align*}
\]

(A1)

where \(\{b_n\}\) is a sequence in \([0, 1]\).

Recently Shahzad and Udomene proved the following results.

**Theorem 1** (Shahzad and Udomene 2006). Let \(K\) be a nonempty closed convex subset of a real uniformly convex Banach space \(E\). Let \(S, T : K \to K\) be two uniformly continuous asymptotically quasi-nonexpansive mappings with sequences \(\{s_n\}, \{t_n\} \subset [1, \infty)\) such that \(\sum_{n=1}^{\infty} (s_n - 1) < \infty\) and \(\sum_{n=1}^{\infty} (t_n - 1) < \infty\), and \(F = F(S) \cap F(T) = \{x \in K : Sx = Tx = x\} \neq \emptyset\). Let \(\{b_n\}\) and \(\{b'_n\}\) be sequences in \([\delta, 1 - \delta]\) for some \(\delta \in (0, 1)\). From arbitrary \(x_1 \in K\), define the sequence \(\{x_n\}\) by the recursion (S2). Then

\[
\lim_{n \to \infty} \|x_n - S^n x_n\| = 0 = \lim_{n \to \infty} \|x_n - T^n x_n\|.
\]

**Theorem 2** (Shahzad and Udomene 2006). Let \(E, K, S, T\) and \(\{x_n\}\) be as in Theorem 1. Assume in addition, that either \(T\) or \(S\) is compact. Then \(\{x_n\}\) converges strongly to some common fixed point of \(S\) and \(T\).

**Theorem 3** (Shahzad and Udomene 2006). Let \(E, K, S, T\) and \(F\) be as in Theorem 2. Let \(\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}\) be sequences in \([0, 1]\) with \(a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n\) for all \(n \geq 1\). From arbitrary \(x_1 \in K\), define the sequence \(\{x_n\}\) by the recursion (S1). Suppose \(\{b_n + c_n\}, \{b'_n + c'_n\} \in [\delta, 1 - \delta]\) for some \(\delta \in (0, 1)\), \(\sum_{n=1}^{\infty} c_n < \infty\) and \(\sum_{n=1}^{\infty} c'_n < \infty\). Then \(\{x_n\}\) converges strongly to some common fixed point of \(S\) and \(T\).
In this paper, we study the weak and strong convergence of iterative scheme given in \((S_1)\) for two strongly asymptotically quasi nonexpansive mappings in a uniformly convex Banach space.

(i) In order to prove our results, we modify the method of proofs, used by Shahzad and Udomene (Shahzad and Udomene 2006) by avoid the conditions \(\{b_n^i\} \in [\delta, 1]\) for some \(\delta > 0\) and \(\{b_n^i + c_n^i\} \in [\delta, 1]\) for some \(\delta \in (0, 1)\), respectively. Thus, in our theorems can be \(b_n^i = 0\) and \(b_n^i + c_n^i = 0\) for all \(n \geq 0\). Therefore, the Mann type convergence theorem can be obtained as corollaries. Similar results for usual Ishikawa iterations (1.2) for one mapping can be obtained, and consequently results of Schu (Schu 1991) can be recovered.

(ii) The same argument can be applied for the results of Fukhar-ud-din and Khan (Fukhar-ud-din and Khan 2007b), (Fukhar-ud-din and Khan 2007a) and Khan et al. (Khan et al. 2008).

(iii) In (Fukhar-ud-din and Khan 2007a), Fukhar-ud-din and Khan discussed the following three-step iteration process for two mappings case:

\[
\begin{align*}
  x_1 & \in C, \\
  x_{n+1} & = a_n x_n + b_n T_1 y_n + c_n u_n, \\
  y_n & = a'_n x_n + b'_n T_2 z_n + c'_n v_n, \\
  z_n & = a''_n x_n + b''_n T_1 x_n + c''_n w_n, \quad n \geq 1,
\end{align*}
\]

and claimed that \("the condition like \(0 < b''_n < 1\) on \(b''_n\) is superfluous so that the iteration scheme \((ADF)\) can be used to approximate the common fixed points under a free parameter\) which is not correct, because for computational purpose \(b''_n\) is involved in \((ADF)\). Also as we already discussed in (i), their claim about Corollary 4.5 ( (Senter and Dotson 1974), page 1359) is not correct.

2 MAIN RESULTS

In this section, we prove weak and strong convergence theorems. We need the following results.

Lemma 4 ( (Tan and Xu 1993)). Let \(\{r_n\}, \{s_n\}, \{t_n\}\) be three nonnegative sequences satisfying the following condition:

\[
r_{n+1} \leq (1 + s_n) r_n + t_n \quad \text{for all} \quad n \in \mathbb{N}.
\]

If \(\sum_{n=1}^{\infty} s_n < \infty\), \(\sum_{n=1}^{\infty} t_n < \infty\), then \(\lim_{n \to \infty} r_n\) exists.

Lemma 5 ( (Górnicki 1989)). Let \(E\) be a uniformly convex Banach space satisfying Opial’s condition and let \(C\) be a nonempty closed convex subset of \(E\). Let \(T\) be asymptotically nonexpansive mapping of \(C\) into itself. Then \(I - T\) is demiclosed with respect to zero.

Lemma 6 ( (Xu 1991)). Let \(p > 1, r > 0\) be two fixed numbers. Then \(E\) is uniformly convex if and only if there exists a continuous, strictly increasing and convex function \(g : [0, \infty) \to [0, \infty)\), \(g(0) = 0\), such that

\[
\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - w_p(\lambda) g (\|x - y\|),
\]

for all \(x, y\) in \(B_r = \{x \in E : \|x\| \leq r\}, \lambda \in [0, 1]\), where \(w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)\).

Note that the inequality in Lemma 6 is known as Xu’s inequality. Now we shall extend Xu’s result. Denote $w_p(\alpha, \beta) = \alpha^p \beta + \alpha \beta^p$.

**Lemma 7** Let $p > 1$, $r > 0$ be two fixed numbers. Then $E$ is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that
\[
\|ax + by + cz\|^p \leq a \|x\|^p + b \|y\|^p + c \|z\|^p
- \max\{w_p(a, b) g (\|x - y\|), w_p(a, c) g (\|x - z\|), w_p(b, c) g (\|y - z\|)\}
\]
for all $x, y, z$ in $B_r(0) = \{x \in E : \|x\| \leq r\}$ and $a, b, c \in [0, 1]$ with $a + b + c = 1$.

**Proof.** If $c = 1$, then $a = 0$, $b = 0$, $w_p(a, b) = 0$, $w_p(a, c) = 0$, $w_p(b, c) = 0$ and so (8) holds. Suppose now that $c < 1$. We first observe that $ax/(1 - c) + by/(1 - c) \in B_r(0)$ for all $x, y$ in $B_r(0)$. It follows from Lemma 3 that
\[
\|ax + by + cz\|^p = \left\|(1 - c) \left(\frac{a}{1 - c} x + \frac{b}{1 - c} y\right) + cz\right\|^p
\leq (1 - c) \left\|\frac{a}{1 - c} x + \frac{b}{1 - c} y\right\|^p + c \|z\|^p
\leq (1 - c) \left[\frac{a}{1 - c} \|x\|^p + \frac{b}{1 - c} \|y\|^p
- w_p\left(\frac{a}{1 - c}\right) g (\|x - y\|)\right] + c \|z\|^p
= a \|x\|^p + b \|y\|^p + c \|z\|^p
- (1 - c) w_p\left(\frac{a}{1 - c}\right) g (\|x - y\|).
\]
Since
\[
w_p\left(\frac{a}{1 - c}\right) = \frac{w_p(a, b)}{(1 - c)^{p+1}} \geq w_p(a, b),
\]
we have
\[
\|ax + by + cz\|^p \leq a \|x\|^p + b \|y\|^p + c \|z\|^p - w_p(a, b) g (\|x - y\|).
\]
Similarly, one can prove (8) with $w_p(a, c)$ and $w_p(b, c)$.

**Remark 8** For $c = 0$, in (8) we obtain the Xu’s inequality.

**Lemma 9** Let $E$ be a normed space and $C$ its nonempty convex subset. Let $S, T : C \to C$ be two strongly asymptotically quasi nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by
\[
x_1 \in C,
x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n, \quad (S_1)
y_n = a'_n x_n + b'_n T^n x_n + c'_n v_n, \quad n \geq 1,
\]
with $\sum_{n=1}^{\infty} (c_n + c'_n) < \infty$. If $F(S) \cap F(T) \neq \emptyset$, then $\lim_{n \to \infty} \|x_n - x^*\| = 0$ for all $x^* \in F(S) \cap F(T)$.

Proof. Assume that

\[ M = \max \{ \sup_{n \geq 1} \| u_n - x^* \|, \sup_{n \geq 1} \| v_n - x^* \| \}, \]

and \( F(S) \cap F(T) \neq \emptyset \). Let \( x^* \in F(S) \cap F(T) \). Then

\[
\| x_{n+1} - x^* \| = \| a_n x_n + b_n S^ny_n + c_n u_n - x^* \|
\leq a_n \| x_n - x^* \| + b_n \| S^ny_n - x^* \| + c_n \| u_n - x^* \|
\leq (1 - b_n) \| x_n - x^* \| + b_n \| S^ny_n - T^nx^* \| + Mc_n
\leq (1 - b_n) \| x_n - x^* \| + b_n k_n \| y_n - x^* \| + Mc_n. \tag{9}
\]

\[
\| y_n - x^* \| = \| a'_n x_n + b'_n T^nx_n + c'_n v_n - x^* \|
\leq a'_n \| x_n - x^* \| + b'_n \| T^nx_n - x^* \| + c'_n \| v_n - x^* \|
\leq (1 - b'_n) \| x_n - x^* \| + b'_n \| T^nx_n - S^nx^* \| + M'c'_n
\leq (1 - b'_n) \| x_n - x^* \| + b'_n k_n \| x_n - x^* \| + M'c'_n
\leq k_n \| x_n - x^* \| + M'c'_n. \tag{10}
\]

Substituting (10) in (9) yields

\[
\| x_{n+1} - x^* \| \leq [1 + (k_n^2 - 1)b_n] \| x_n - x^* \| + M(k_n b_n c'_n + c_n)
\leq [1 + (k_n^2 - 1)] \| x_n - x^* \| + M(k_n b_n c'_n + c_n).
\]

As an application of the Lagrange mean value theorem for the function \( f(x) = x^q \), we can see that for \( x \in [1, 2] \) and \( q > 1 \)

\[ x^q - 1 = qe^{q-1}(x-1) \leq q^2(x-1). \]

This together with the assumption \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \) implies that \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \). By Lemma 4, \( \lim_{n \to \infty} \| x_n - x^* \| \) exists for each \( x^* \in F(S) \cap F(T) \). \( \blacksquare \)

Theorem 10 Let \( E \) be a uniformly convex Banach space and \( C \) be a nonempty closed convex subset of \( E \). Let \( S, T : C \to C \) be two strongly asymptotically quasi nonexpansive mappings satisfying \( \| S^n x - T^ny \| \leq k_n \| x - y \| \) for all x, y \( \in C \), \( n \geq 1 \), and \( \{ x_n \} \) be the sequence as defined in (S1) with \( \{ b_n \} \subset [\delta, 1-\delta] \) for some \( \delta \in (0, 1) \), \( \limsup_{n \to \infty} b'_n < 1 \) and \( \sum_{n=1}^{\infty} (c_n + c'_n) < \infty \). If \( F(S) \cap F(T) \neq \emptyset \), then

\[ \lim_{n \to \infty} \| S x_n - x_n \| = 0 = \lim_{n \to \infty} \| T x_n - x_n \|. \]

Proof. Since by Lemma 9, \( \lim_{n \to \infty} \| x_n - x^* \| \) exists for all \( x^* \in F(S) \cap F(T) \), then \( \{ x_n - x^* \} \) is bounded. Put

\[ M_1 = \max \{ \sup_{n \geq 1} \| x_n - x^* \|, \sup_{n \geq 1} \| u_n - x^* \|, \sup_{n \geq 1} \| v_n - x^* \| \}. \]

\( \text{Int. J. of Appl. Math. and Mech. 6(4): 68-81, 2010.} \)
By Lemma 7, we have

\[
\|y_n - x^*\|^p = \|a_n x_n + b_n^T x_n + c_n v_n - x^*\|^p \\
= \|a_n'(x_n - x^*) + b_n'(T x_n - x^*) + c_n'(v_n - x^*)\|^p \\
\leq a_n'\|x_n - x^*\|^p + b_n'\|T x_n - x^*\|^p + c_n'\|v_n - x^*\|^p \\
- w_p(a_n', b_n')g(\|x_n - T x_n\|) \\
\leq (1 - b_n')\|x_n - x^*\|^p + b_n' k_n^p\|x_n - x^*\|^p + M_p^p c_n \\
= (1 + (k_n^p - 1) b_n')\|x_n - x^*\|^p + M_p^p c_n \\
\leq k_n^p\|x_n - x^*\|^p + M_p^p c_n.
\] (11)

Also

\[
\|x_{n+1} - x^*\|^p = \|a_n x_n + b_n S^y y_n + c_n u_n - x^*\|^p \\
= \|a_n (x_n - x^*) + b_n (T^y y_n - x^*) + c_n (u_n - x_n)\|^p \\
\leq a_n \|x_n - x^*\|^p + b_n\|S^y y_n - x^*\|^p + c_n\|u_n - x^*\|^p \\
- w_p(a_n, b_n)g(\|x_n - S^y y_n\|) \\
\leq (1 - b_n)\|x_n - x^*\|^p + b_n k_n^p\|y_n - x^*\|^p + M_p^p c_n \\
- w_p(a_n, b_n)g(\|x_n - S^y y_n\|).
\] (12)

Substituting (11) in (12), we obtain

\[
\|x_{n+1} - x^*\|^p \leq (1 + (k_n^p - 1) b_n)\|x_n - x^*\|^p + M_p^p k_n^p b_n c_n + M_p^p c_n \\
- w_p(a_n, b_n)g(\|x_n - S^y y_n\|) \\
\leq \|x_n - x^*\|^p + M_p^p [(1 - \delta)(k_n^p - 1) + (1 - \delta) k_n^p c_n + c_n] \\
- w_p(a_n, b_n)g(\|x_n - S^y y_n\|).
\] (13)

By \( \lim c_n = 0 \), there exists a natural number \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), we have \( c_n \leq \delta - \eta; \eta \in (0, \delta) \). Now by \( \{b_n\}_{n \geq 1} \subset [\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \) and \( c_n \leq \delta - \eta; \eta \in (0, \delta) \), we have \( w_p(a_n, b_n) \geq (\delta - \eta)\delta + (\delta - \eta)\delta^p = \Omega_{p, \delta, \eta} \). Hence and form (13), we get

\[
\|x_{n+1} - x^*\|^p \leq \|x_n - x^*\|^p + M_p^p [(1 - \delta)(k_n^p - 1) + (1 - \delta) k_n^p c_n + c_n] \\
- \Omega_{p, \delta, \eta}g(\|x_n - S^y y_n\|).
\]

Hence

\[
\Omega_{p, \delta, \eta}g(\|x_n - S^y y_n\|) \leq \|x_n - x^*\|^p - \|x_{n+1} - x^*\|^p + M_p^p \gamma_n, \] (14)

where \( \gamma_n = (1 - \delta)(k_n^p - 1) + (1 - \delta) k_n^p c_n + c_n \).

It follows from inequality (14) that for any natural number \( m > N_0 \),

\[
\Omega_{p, \delta, \eta} \sum_{n=N_0}^{m} g(\|x_n - S^y y_n\|) \leq \|x_{N_0} - x^*\|^2 - \|x_{m+1} - x^*\|^2 + \sum_{n=N_0}^{m} \gamma_n \\
\leq \|x_{N_0} - x^*\|^2 + \sum_{n=N_0}^{m} \gamma_n. \] (15)

As in the proof of Lemma 9, we know that the assumption $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty} (k_n^q - 1) < \infty$. Taking the limit of both sides of (15) as $n \to \infty$ yields

$$\sum_{n=N_0}^{\infty} g(\|x_n - S^ny_n\|) < \infty.$$ 

Therefore $\lim_{n \to \infty} g(\|x_n - S^ny_n\|) = 0$. Since $g$ is strictly increasing and continuous at 0, it follows that

$$\lim_{n \to \infty} \|x_n - S^ny_n\| = 0. \quad (16)$$

Next consider

$$\|x_n - T^nx_n\| \leq \|x_n - S^ny_n\| + \|S^ny_n - T^nx_n\| \leq \|x_n - S^ny_n\| + k_n \|y_n - x_n\|. \quad (17)$$

$$\|y_n - x_n\| = \|a_n'x_n + b_n'T^nx_n + c_n'y_n - x_n\| \leq b_n' \|x_n - T^nx_n\| + c_n' \|y_n - x_n\| \leq b_n' \|x_n - T^nx_n\| + 2M_1 c_n'. \quad (18)$$

Substituting (18) in (17), we get

$$\|x_n - T^nx_n\| \leq \|x_n - S^ny_n\| + b_n'k_n \|x_n - T^nx_n\| + 2M_1 c_n'k_n,$$

implies

$$\|x_n - T^nx_n\| \leq \frac{1}{1 - b_n'k_n} \|x_n - S^ny_n\| + 2M_1 \frac{c_n'k_n}{1 - b_n'k_n},$$

gives us with the help of condition $\lim_{n \to \infty} \sup b_n' < 1$,

$$\lim_{n \to \infty} \|x_n - T^nx_n\| = 0. \quad (19)$$

Let $d_n = \|x_n - S^ny_n\|$ and $e_n = \|x_n - T^nx_n\|$. Now observe that

$$\|x_{n+1} - Sx_{n+1}\| \leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Sx_{n+1}\| \leq e_{n+1} + k_1 \|T^n x_{n+1} - x_{n+1}\| \leq e_{n+1} + k_1 (\|T^n x_{n+1} - S^ny_n\| + \|S^ny_n - x_n\| + \|x_n - x_{n+1}\|) \leq e_{n+1} + k_1 (1 + k_n) \|x_{n+1} - x_n\| + k_n \|x_n - y_n\| + d_n. \quad (20)$$

However

$$\|x_n - x_{n+1}\| = \|x_n - a_n x_n - b_n S^ny_n - c_n u_n\| \leq b_n \|x_n - S^ny_n\| + c_n \|u_n - x_n\| \leq b_n d_n + 2M_1 c_n, \quad (21)$$

and

\[ \|x_n - y_n\| = \|x_n - a'_{n}x_n - b'_nT^m\| + c'_n v_n\|
\leq b'_n \|x_n - T^m x_n\| + c'_n \|v_n - x_n\|
\leq b'_ne_n + 2M_1 c'_n. \tag{22} \]

Substituting (21) and (22) in (20) yields

\[ \|x_{n+1} - Sx_{n+1}\| \leq e_{n+1} + k_1 \{(1 + k_n)b_n + 1\} d_n
+ k_n b'_n e_n + 2M_1 [(1 + k_n)c_n + k_n c'_n] , \]

implies

\[ \lim_{n \to \infty} \|x_{n+1} - Sx_{n+1}\| = 0. \]

Thus

\[ \lim_{n \to \infty} \|x_n - Sx_n\| = 0. \]

Also

\[ \|x_{n+1} - Tx_{n+1}\| \leq \|x_{n+1} - x_n\| + \|x_n - Sx_n\| + \|Sx_n - Tx_{n+1}\|
\leq \|x_{n+1} - x_n\| + \|x_n - Sx_n\| + k_1 \|x_n - x_{n+1}\|
= (1 + k_1) \|x_n - x_{n+1}\| + \|x_n - Sx_n\|
\leq (1 + k_1) (b_n d_n + 2M_1 c_n) + \|x_n - Sx_n\| , \]

implies

\[ \lim_{n \to \infty} \|x_{n+1} - Tx_{n+1}\| = 0. \]

Thus

\[ \lim_{n \to \infty} \|x_n - Tx_n\| = 0. \]

Hence

\[ \lim_{n \to \infty} \|Sx_n - x_n\| = 0 = \lim_{n \to \infty} \|Tx_n - x_n\|. \]

This completes the proof of Theorem. \( \blacksquare \)

**Theorem 11** Let \( E \) be a uniformly convex Banach space satisfying the Opial’s condition and \( C, S, T \) and \( \{x_n\} \) be as in Theorem 10. If \( F(S) \cap F(T) \neq \emptyset \), then \( \{x_n\} \) converges weakly to a common fixed point of \( S \) and \( T \).

Proof. Let \( x^* \in F(S) \cap F(T) \). Then, as proved in Lemma 7, \( \lim_{n \to \infty} \|x_n - x^*\| \) exists. Now we prove that \( \{x_n\} \) has a unique weak cluster point limit in \( F(S) \cap F(T) \). To prove this, let \( z_1 \) and \( z_2 \) be weak limits of the subsequences \( \{x_{n_1}\} \) and \( \{x_{n_2}\} \) of \( \{x_n\} \), respectively. By Theorem 10, \( \lim_{n \to \infty} \|x_n - Sx_n\| = 0 \) and by Lemma 5, \( I - S \) is demiclosed with respect to zero. Therefore, we obtain \( Sz_1 = z_1 \). Similarly, \( Tz_1 = z_1 \). Again in the same way, we can prove that \( z_2 \in F(S) \cap F(T) \). Next, we prove the uniqueness. Suppose, to the contrary, that \( z_1 \neq z_2 \). Then by the Opial’s condition we have

\[
\lim_{n \to \infty} \|x_n - z_1\| = \lim_{n \to \infty} \|x_{n_1} - z_1\| < \lim_{n \to \infty} \|x_{n_1} - z_2\| = \lim_{n \to \infty} \|x_n - z_2\| = \lim_{n \to \infty} \|x_{n_j} - z_2\| < \lim_{n \to \infty} \|x_{n_j} - z_1\| = \lim_{n \to \infty} \|x_n - z_1\|,
\]

a contradiction. Hence \( \{x_n\} \) converges weakly to a point in \( F(S) \cap F(T) \). ■

The following condition is due to Senter and Dotson (Senter and Dotson 1974).

**Definition 9.** A mapping \( T : C \to C \) where \( C \) is a subset of \( E \), is said to satisfy the condition (A) if there exists a nondecreasing function \( f : [0, \infty) \to [0, \infty) \) with \( f(0) = 0 \), \( f(r) > 0 \) for all \( r \in (0, \infty) \) such that

\[
\|x - Tx\| \geq f(d(x, F(T))) \text{ for all } x \in C, \tag{A}
\]

where \( d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\} \).

Senter and Dotson (Senter and Dotson 1974) approximated fixed points of a nonexpansive mapping \( T \) by Mann iterates. Later on, Maiti and Ghosh (Maiti and Ghosh 1989) and Tan and Xu (Tan and Xu 1993) studied the approximation of fixed points of a nonexpansive mapping \( T \) by Ishikawa iterates under the same condition (A) which is weaker than the requirement that \( T \) is demicompact.

We modify the condition (A) for two mappings \( S, T : C \to C \) as follows:

**Definition 10.** Two mappings \( S, T : C \to C \) where \( C \) a subset of \( E \), are said to satisfy the condition (AR) if there exists a nondecreasing function \( f : [0, \infty) \to [0, \infty) \) with \( f(0) = 0 \), \( f(r) > 0 \) for all \( r \in (0, \infty) \) and \( \lambda \in [0, 1] \) such that

\[
\lambda \|x - Tx\| + (1 - \lambda) \|x - Sx\| \geq f(d(x, F)) \text{ for all } x \in C, \tag{AR}
\]

where \( d(x, F) = \inf\{\|x - x^*\| : x^* \in F = F(S) \cap F(T)\} \).

Note that the condition (AR) reduces to condition (A) when \( S = T \). We shall use condition (AR) instead of compactness to study the strong convergence of \( \{x_n\} \) defined in (S1). It is worth noting that in case of two asymptotically quasi nonexpansive mappings \( S, T : C \to C \), condition (AR) is weaker than the compactness.

Next, we approximate the common fixed points using (S1) by the following strong convergence theorem.

Theorem 12 Let $E$ be a uniformly convex Banach space and $C, S, T$ and $\{x_n\}$ be as in Theorem 10. Further, let $S$ and $T$ satisfy condition (AR). If $F(S) \cap F(T) \neq \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.

Proof. By Lemma 9, $\lim_{n \to \infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$. Let it be $c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. By Theorem 10, $\lim_{n \to \infty} \|Sx_n - x_n\| = 0 = \lim_{n \to \infty} \|Tx_n - x_n\|$. As proved in Lemma 9, we have

$$\|x_{n+1} - x^*\| \leq 1 + (k_n^2 - 1) \|x_n - x^*\| + M(k_n b_n c_n + c_n),$$

gives that $\lim_{n \to \infty} d(x_n, F) = 0$ by virtue of Lemma 4. Now by condition (AR), $\lim_{n \to \infty} f(d(x_n, F)) = 0$. Since $f$ is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n \to \infty} d(x_n, F) = 0$. The rest of the proof follows the lines similar to Tan and Xu (Tan and Xu 1993) and is therefore omitted. ■

By specifying the corresponding parameters in our lemmas and theorems we obtain the following results.

Corollary 13 (Senter and Dotson 1974). Let $E$ be a uniformly convex Banach space and $C$ a nonempty closed and convex subset of $E$. Let $T$ be a nonexpansive self-mapping of $C$ satisfying the condition (A). Define a sequence $\{x_n\}$ in $C$ as: $x_1 \in C$, $x_{n+1} = (1 - a_n)x_n + a_nTx_n$, $n \in \mathbb{N}$, where $\{a_n\}$ is a sequence in $[a, b]$ such that $0 < a < b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.

Lemma 14 Let $E$ be a normed space and $C$ its nonempty convex subset. Let $S, T : C \to C$ be two strongly asymptotically quasi nonexpansive mappings and $\{x_n\}$ be the sequence as defined in $(S_2)$. If $F(S) \cap F(T) \neq \emptyset$, then $\lim_{n \to \infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.

Lemma 15 Let $E$ be a uniformly convex Banach space and $C$ its nonempty closed convex subset. Let $S, T : C \to C$ be two strongly asymptotically quasi nonexpansive mappings satisfying $\|T^nx - T^ny\| \leq k_n \|x - y\|$ for all $x, y \in C, n \geq 1$, and $\{x_n\}$ be the sequence as defined in $(S_2)$ with $\{b_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ and $\lim_{n \to \infty} b_n' < 1$. If $F(S) \cap F(T) \neq \emptyset$, then $\lim_{n \to \infty} \|Sx_n - x_n\| = 0 = \lim_{n \to \infty} \|Tx_n - x_n\|$.

Theorem 16 Let $E$ be a uniformly convex Banach space satisfying the Opial’s condition and $C, S, T$ and $\{x_n\}$ be as taken in Lemma 15. If $F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of $S$ and $T$.

Theorem 17 Let $E$ be a uniformly convex Banach space and $C, S, T$ and $\{x_n\}$ be as taken in Lemma 15. Further, let $S$ and $T$ satisfy condition (AR). If $F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.

Lemma 18 Let $E$ be a normed space and $C$ its nonempty convex subset. Let $T : C \to C$ be an asymptotically nonexpansive mapping and $\{x_n\}$ be the sequence as defined in $(A_1)$. If $F(T) \neq \emptyset$, then $\lim_{n \to \infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$.
Lemma 19 Let $E$ be a uniformly convex Banach space and $C$ its nonempty closed convex subset. Let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping and $\{x_n\}$ be the sequence as defined in $(A_1)$ with $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If $F(T) \neq \emptyset$, then $\lim_{n \to \infty} \|Tx_n - x_n\| = 0.$

Theorem 20 Let $E$ be a uniformly convex Banach space satisfying the Opial’s condition and $C, T$ and $\{x_n\}$ be as taken in Lemma 19. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of $T$.

Theorem 21 Let $E$ be a uniformly convex Banach space and $C, T$ and $\{x_n\}$ be as taken in Lemma 19. Further let $T$ is satisfying condition $(A)$. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of $T$.

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