ANALYTICAL SOLUTIONS TO THE THREE-DIMENSIONAL INCOMPRESSIBLE FLOWS THROUGH THE VORTICITY EQUATIONS

Gunawan Nugroho¹, Ahmed M. S. Ali², Zainal A. Abdul Karim³, Wahyu D. H. Wijayanti⁴

¹,⁴Department of Engineering Physics, Institut Teknologi Sepuluh Nopember, Jl Arief Rahman Hakim, Surabaya, Indonesia (60111)
Email: gunawan@ep.its.ac.id, gunawanzz@yahoo.com, hapsari.devi04@gmail.com
²Department of Mechanical Engineering, The University of British Columbia, Vancouver BC, Canada
Email: a.m.s.ali@alumni.lboro.ac.uk
³Department of Mechanical Engineering, Universiti Teknologi Petronas, Bandar Seri Iskandar 31750, Tronoh, Perak, Malaysia
Email: ambri@petronas.com.my

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ABSTRACT

The three-dimensional incompressible flows described by Navier-Stokes equations and the continuity equation are solved in this work with the application of the transformed coordinate \( \xi = kz - \zeta(t) \). The solution is proposed to be in the form \( \mathbf{v} = \nabla \Phi + \nabla \times \Phi \) where \( \Phi \) is a potential function that is defined as \( \Phi = P(x, y, \xi)R(y)S(\xi) \). The potential function is firstly substituted into the continuity equation to produce the solution for \( R \) and \( S \). The resultant expression is used sequentially in the vorticity equations to reduce the problem to the class of nonlinear ordinary differential equations in \( P \) terms. General solutions are obtained based on the particular solutions of \( P \) by implementing a novel technique. Finally, the pressure is solved by applying the velocity vector into the Navier-Stokes equations to complete the solutions. The uniqueness of the solution is ensured by generating and equating two solutions. Moreover, the solution is regularized for blow up cases with a controllable error. Further analysis shows that the energy rate is not zero for any nontrivial solution with respect to initial and boundary conditions. The solution being nontrivial represents the qualitative nature of turbulent flows. The simulation shows the fluctuation of the decaying velocity through time due to energy dissipation. The rapid velocity accumulations are also detected providing that the solutions may produce singularity in the small scale of turbulent flows. The bifurcation is then detected which revealed the strong nonlinearity in the small scale of turbulent flows. It is concluded that the selection of variables for the potential function can be interchanged from the beginning, resulting in similar explicit solutions.

Keywords: Continuity equation, the Navier-Stokes Equations, potential function, analytical solutions, vorticity equations, partial differential equations
1 INTRODUCTION

The main difficulty of solving the Navier-Stokes equations exactly is the contribution of the nonlinear terms representing fluid inertia which then troubled the conventional analysis in general cases. Thus, it is not surprising that there exists only a few analytical solutions until recently, and the full solution of the three-dimensional Navier-Stokes equations still remains as one of the open problems in mathematical physics. Even the qualitative behaviour in term of existence and uniqueness theorem for global smooth solution, is not settled yet (Ohkitani, 2008).

The problem of searching for the classes of analytical solutions of the full Navier-Stokes equations is highly demanding from a practical viewpoint, as has been described in the literature (Kao, 1980). Analytical solutions often play a special role in the theory of nonlinear equations and they are found able to describe the detailed behaviour of the concerning systems (Galaktionov & Svirschchevskii, 2007). Analytical solutions also facilitate a theoretical understanding, paving the way to global solutions. They may help explain the issue of global smoothness in time (Zhou, 2006). Moreover, the solutions may be examined as models for turbulence (Ohkitani, 2004). Also, some particular solutions such as vortex solutions play a significant role in the development of turbulence theories (Lydberg & Tryggeson, 2007).

Unfortunately, only a few analytical works are present in the literatures (Wang, 1989, Wang, 1991, Okamoto, 2003). As in the most cases, analytical solutions are examined only in special conditions in which the nonlinearity are weakened or even removed from the analysis. The type of the simplified analysis is applicable for steady and unsteady Couette and Poiseuille flows in which the nonlinear terms are removed permanently (White, 2003). The other less known example is applicable to Beltrami flows in which the nonlinear terms are nonzero in the Navier-Stokes equations but they fade in the vorticity equations (Shapiro, 1993).

However, more sophisticated analysis of the Navier-Stokes equations is also conducted and gives more insight to the problems. One of them is the transformation of the Navier-Stokes equations to the Schrodinger equation, performed by application of the Riccati equation (Christianto & Smaranche, 2008). It has good prospects since the Schrodinger equation is linear and has well defined solutions. The method of Lie group theory is also applied in order to transform the original partial differential equations into ordinary differential systems (Kamran et al., 2006). It is concluded that an approximate series solution is obtained. The same route is taken by Meleshko (2004) and by Thailert (2005), in transforming the Navier-Stokes equations to solvable linear systems. Furthermore, less popular methods, such as the Hodograph-Legendre transformation, have also been applied to reduce the original problem to one more tractable, and thus closer to the goal of obtaining analytical solutions (Mohyuddin et al., 2008). The reduction of the full set of Navier-Stokes equations to be a class of nonlinear ordinary differential equation is also performed (Nugroho et al., 2009). The solution applied to both zero and constant pressure gradient cases. The method of introducing special solutions for velocity has also been investigated in (Sidorov, 1989, Rajagopal, 1984).

The vorticity representation is reasonable and physically clear, at least for incompressible flows (Chorin, 2002, Majda & Bertozzi, 1999). It is interesting to mention that solutions of the vorticity equations drive towards the collection of analytical solutions to the Navier-Stokes equations. Therefore, any analytical solution to the vorticity equations will be of a practical value.
This work is a continuation of our previous work on a special of analytical solutions. In this work, a special class of solutions is introduced and applied to the vorticity equations. A potential function and a transformed coordinate are proposed, and the three vorticity equations are altered into simpler equations in terms of the potential function and the transformed coordinates. Unlike the previous work, a nontrivial coordinate relation with respect to time is implemented in this work. The proposed class of solutions is substituted firstly in the continuity equation and the resultant expression is employed sequentially into the vorticity system to find full solutions. Then, the analytical solutions are obtained and extended to the more general form. Furthermore, the analytical solutions of the vorticity equations are then substituted into the Navier-Stokes equations in order to obtain the pressure solutions.

2 THE MAIN THEOREM

The three-dimensional incompressible Navier-Stokes equations and the continuity equation are,

\[ \frac{\partial V}{\partial t} + V \nabla V = -\frac{1}{\rho} \nabla p + \nu \nabla^2 V \]  

(1a)

\[ \nabla \cdot V = 0 \]  

(1b)

where \( p \) is static pressure, \( \rho \) is fluid density, \( \nu \) is kinematic viscosity and \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \). All solutions describe the three velocity components in the three spatial directions, i.e., \( V = (u, v, w) \), \( u = u(x, y, z, t) \), \( v = v(x, y, z, t) \) and \( w = w(x, y, z, t) \). Consider a potential function \( \Phi \), so that the velocity components are the derivatives of the function and can be expressed as,

\[ V = \nabla \Phi + \nabla \times \Phi \]  

(2a)

Therefore, the velocity components are expressed as follows,

\[ u = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} - k \frac{\partial \Phi}{\partial \xi}, \quad v = \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial x}, \quad \text{and} \quad w = \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial y} \]  

(2b)

The coordinate transformation is applied,

\[ \xi = k z - \zeta(t) \]  

(2c)

The above transformation is the extension of that given by Mohyuddin et. al. (2008) and Nugroho et al. (2009). The previous work describes \( \xi \) as a combination of two spatial coordinate with linear variation of time. The function described in (2c) provides functional variation of time \( t \) which is more general. Then, velocity components in equation (2b) can now be rewritten using the new coordinate as,

\[ u = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} - k \frac{\partial \Phi}{\partial \xi}, \quad v = \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial x}, \quad \text{and} \quad w = k \frac{\partial \Phi}{\partial \xi} + \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial y} \]  

(2d)
However, it is reasonable to investigate the Navier-Stokes equations in terms of vorticity equations (Chae & Choe, 1999, Kozono & Yatsu, 2004). The second term of (2a) will contribute to the rotational nature of the flows and may be related to the vorticity in the system or generated at the boundary. Vorticity is a flow parameter which does not propagate instantly, this is a main reason of seeing vorticity as a fundamental quantity of fluid flows. Taking curl operation to the Navier Stokes equations, the following vorticity equations are obtained,

\[
\frac{\partial \omega}{\partial t} + V \cdot \nabla \omega = \omega \cdot \nabla V + \nu \nabla^2 \omega
\]  

with \( \omega = \nabla \times V \) and \( \omega = (\omega_x, \omega_y, \omega_z) \). The vorticity components are distributed in three dimensions, \( \omega_x = \omega_x(x,y,z,t) \), \( \omega_y = \omega_y(x,y,z,t) \) and \( \omega_z = \omega_z(x,y,z,t) \). Note that the pressure term in (1a) is vanished by curl procedure. It is not hard to see that vorticity components will satisfy,

\[
\omega_x = \frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Phi}{\partial y \partial \xi} - k^2 \frac{\partial^2 \Phi}{\partial \xi^2} + k \frac{\partial^2 \Phi}{\partial x \partial \xi},
\]

\[
\omega_y = k \frac{\partial^2 \Phi}{\partial y \partial \xi} - k^2 \frac{\partial^2 \Phi}{\partial \xi^2} - \frac{\partial^2 \Phi}{\partial x \partial \xi},
\]

and

\[
\omega_z = k \frac{\partial^2 \Phi}{\partial x \partial \xi} - \frac{\partial^2 \Phi}{\partial x \partial \xi} - k \frac{\partial^2 \Phi}{\partial y \partial \xi}
\]

(3b)

The potential function is assumed to take the following particular form, which will satisfy the continuity and vorticity equations,

\[
\Phi = P(x,y,\xi)R(y)S(\xi)
\]

(4)

Therefore, the following theorem is produced from the problem statement above,

Theorem 1: Given \( V \) is a velocity vector that satisfies the continuity and the vorticity equations over \( x, y \) and \( \xi \), where the transformed coordinate \( \xi \) is defined as \( \xi = kz - \zeta(t) \), where \( k \) is a constant. The velocity vector is proposed to be in the form \( V = \nabla \Phi + \nabla \times \Phi \), where the potential function \( \Phi \) is defined as a product of \( P(x,y,\xi), R(y) \) and \( S(\xi) \). Then there exist \( U(x,y,\xi) \) and \( W(x,y,\xi) \) as particular solutions for the reduced vorticity equation

\[
a_3P_{xxx} + b_3P_{xxx}P + c_3P_{xxx}P + d_3P_{xxx}P + e_3P_{xxx}P + f_3P_{xxx}P + g_3P_{xxx}P + h_3P_x^2 + i_3P_xP + j_3P_x = 0
\]

and

\[
R = e^{-\int \left[ \int [C_0(x,y)] \right] dy} \int_y C_0 e^{-\int \left[ \int [C_0(x,y)] \right] dy} dy
\]

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\[ S = e^{-\int_{\xi}^{1} \left( \int C_{1}(\xi) \xi \right) d\xi} \int_{\xi}^{1} C_{2} e^{\int_{\xi}^{1} \left( \int C_{1}(\xi) \xi \right) d\xi} d\xi \]

as general solutions for the reduced continuity equation

\[ R_{x} + \left( \int_{y} C_{2}(y) dy \right) = C_{3} \quad \text{and} \quad S_{\xi} + \left( \int_{\xi} C_{4}(\xi) d\xi \right) = C_{5} \]

They form a potential function as

\[ \Phi = \left\{ U(x, y, \xi) + W(x, y, \xi) + \ldots \right\} e^{-\int_{\xi}^{1} \left( \int C_{1}(\xi) \xi \right) d\xi} \int_{\xi}^{1} C_{2} e^{\int_{\xi}^{1} \left( \int C_{1}(\xi) \xi \right) d\xi} d\xi \]

where \( C_{3} \) and \( C_{5} \) are constants and \( a_{1}, b_{3}, c_{3}, d_{3}, e_{3}, f_{3}, g_{3}, h_{3}, i_{3} \), and \( j_{3} \) are constants with respect to x axis. The potential function then generates the velocity vector such that there exists a static pressure \( p \) which fulfills the following Navier-Stokes equations

\[ \frac{\partial V}{\partial t} + V \nabla V = -\frac{1}{\rho} \nabla p + \nu \nabla^{2} V \]

where \( \rho \) is a constant fluid density. The resulting velocity vector \( V \) and static pressure \( p \) appear as the solutions of the continuity and the three-dimensional incompressible Navier-Stokes equations.

Proof of theorem 1:

Substituting equation (4) into continuity equation will give the following expression,

\[ R_{x} + R_{y} + P_{R_{x}} + k^{2} S_{R_{x}} = P_{R_{y}} + 2P_{y} R_{y} + S + k^{2} P_{S_{y}} + 2k^{2} P_{S_{y}} R_{y} = 0 \]  (5)

Lemma 1: Let \( P, R \) and \( S \) in (5) are separable, then \( R \) and \( S \) are becoming solutions of the first order linear differential equations as

\[ R = e^{-\int_{\xi}^{1} \left( \int C_{1}(\xi) \xi \right) d\xi} \int_{\xi}^{1} C_{2} e^{\int_{\xi}^{1} \left( \int C_{1}(\xi) \xi \right) d\xi} d\xi \quad \text{and} \quad S = e^{-\int_{\xi}^{1} \left( \int C_{4}(\xi) \xi \right) d\xi} \int_{\xi}^{1} C_{5} e^{\int_{\xi}^{1} \left( \int C_{4}(\xi) \xi \right) d\xi} d\xi \]

where \( C_{3} \) and \( C_{5} \) are constants.

Proof: Dividing equation (5) by \( PR_{y} \) and rearranging will produce the following equation,

\[ \frac{P_{x}}{P} + \frac{P_{y}}{P} + k^{2} \frac{P_{z}}{P} = -\frac{R_{y}}{R} - 2 \frac{P_{y} R_{y}}{PR_{y}} = k^{2} \frac{S_{z}}{S} + 2k^{2} \frac{S_{y}}{PS_{y}} + C_{3}(y, \xi) = C_{3}(y) \]  (6a)

Taking, the first relation in the right hand side,

\[ R_{yy} + 2 \frac{P_x}{P} R_y + C_2(y) R = 0 \quad (6b) \]

Let \( \left( 2 \frac{P_x}{P} \right)_y = C_2(y) \) then (6b) can be written as,

\[ R_{yy} + \left[ \int_y C_2(y) dy \right] R = 0 \quad (6c) \]

Integrating the above equation once to yield the first order relation,

\[ R_y + \left[ \int_y C_2(y) dy \right] R = C_3 \quad (6d) \]

The next step is taking the second relation of (6a) as,

\[ S_{\xi\xi} + 2 \frac{P_x}{P} S_\xi + C_4(\xi) S = 0 \quad (6e) \]

where \( C_4(\xi) \) is taken as \( \frac{C_1(y, \xi) - C_2(y)}{k^2} \). By the same procedure, the above equation is reduced into,

\[ S_\xi + \left[ \int_\xi C_4(\xi) d\xi \right] S = C_5 \quad (6f) \]

Therefore, the general solutions for \( R \) and \( S \) can be written as (Coddington, 1989),

\[ R = e^{-\int \left[ \int C_1(y, \xi) d\xi \right] dy} \int_y C_3 e^{\int \left[ \int C_2(y, \xi) d\xi \right] dy} dy, \text{ and } S = e^{-\int \left[ \int C_4(\xi) d\xi \right] d\xi} \int_\xi C_5 e^{\int \left[ \int C_4(\xi) d\xi \right] d\xi} d\xi \quad (6g) \]

where \( C_3 \) and \( C_5 \) are integration constants. This proves lemma 1.

Thus, \( R \) and \( S \) can be substituted to the potential function (4) to produce the following statement,

**Lemma 2:** Let \( \Phi \) be a differentiable potential function that is defined as a product of \( P(x, y, \xi) \), \( R(y) \) and \( S(\xi) \) that relates the velocity vector as \( V = \nabla \Phi + \nabla \times \Phi \) over \( x \), \( y \) and \( \xi \), where \( \xi \) is transformed coordinate defined in (2c). The potential function satisfies continuity equation and reduces the Vorticity equations in the following form

\[ a_3 P_{xxx} + b_3 P_{xxT} + c_3 P_{xx} P + d_3 P_{xx} P + e_3 P_{xx} P + f_3 P_{xx} P + g_3 P_{xx} P + h_3 P^2 + i_3 P x + j_3 P = 0 \]

where \( a_3, b_3, c_3, d_3, e_3, f_3, g_3, h_3, i_3 \) and \( j_3 \) are constants with respect to the \( x \) axis.

**Proof:** Furthermore, the derivation is to apply equation (3b) into vorticity equations in \( x \) component,
The potential function (4) is substituted in the above equation, and the equation can be rewritten as,

\[ a_0 P_{xxx} + b_0 P_{xx} P_x + c_0 P_{xx} P + d_0 P_{xx} + e_0 P_x^2 + f_0 P_x P + g_0 P_x + h_0 P^2 + i_0 P = 0 \]  

(7b)

The next step now is to repeat the procedure applied to the \( x \) component of the vorticity equation, but this time to the vorticity in the \( y \) component, will yield the following,

\[ a_1 P_{xxx} + b_1 P_{xx} P_x + c_1 P_{xx} P + d_1 P_{xx} + e_1 P_{xx} P_x + f_1 P_{xx} P + g_1 P_{xx} + h_1 P_x^2 + i_1 P_x P + j_1 P_x + l_1 P^2 + m_1 P = 0 \]  

(8)

The same procedure is applied to the \( z \) component of the vorticity equation, giving,

\[ a_2 P_{xxx} + b_2 P_{xx} P_x + c_2 P_{xx} P + d_2 P_{xx} + e_2 P_{xx} P_x + f_2 P_{xx} P + g_2 P_{xx} + h_2 P_x^2 + i_2 P_x P + j_2 P_x + l_2 P^2 + m_2 P = 0 \]  

(9)

Substituting equations (7b) and (8) into (9) to eliminate \( P^2 \) and \( P \) as follows,

\[ a_3 P_{xxx} + b_3 P_{xx} P_x + c_3 P_{xx} P + d_3 P_{xx} + e_3 P_{xx} P_x + f_3 P_{xx} P + g_3 P_{xx} + h_3 P_x^2 + i_3 P_x P + j_3 P_x = 0 \]  

(10)

It is noticed that \( a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i, l_i \) and \( m_i \) (where the subscript \( i \) is the constant index) are some constants with respect to the \( x \) axis, but several are \( y \) and \( \xi \) dependent as they are solution of continuity equation. This proves lemma 2.

Letting \( U \) be the known particular solution of (10) and \( W \) be the other solution will generate a more general solution for (10) in the following form,

\[ P = U + W \]  

(11)

Note that the situation is almost hopeless if the general solution is taken as a product of two respective particular solutions i.e. \( P = U W \). Therefore, based on (11), equation (10) is decomposed by substitution into,

\[ a_i U_{xxx} + b_i U_{xx} U_x + b_i U_{xx} W_x + b_i U_{xxx} W + b_i W_{xxx} W + b_i W_{xx} W + b_i U_{xxx} W + b_i U_{xx} W + c_i U_{xxx} W + c_i U_{xx} W + c_i W_{xxx} W + c_i W_{xx} W + d_i U_{xxx} W + d_i W_{xxx} W + e_i U_{xxx} W + e_i U_{xx} W + e_i U_{xxx} W + e_i W_{xxx} W + e_i W_{xx} W + f_i U_{xxx} W + f_i W_{xxx} W + f_i W_{xx} W + f_i U_{xx} W + f_i W_{xx} W + f_i U_{xx} W + f_i W_{xx} W + g_i U_{xxx} W + g_i U_{xx} W + g_i W_{xxx} W + g_i W_{xx} W + h_i U_{xxx} W + h_i W_{xxx} W + i_i U_{xxx} W + i_i W_{xxx} W + i_i W_{xx} W + i_i U_{xx} W + i_i W_{xx} W + i_i U_{xx} W + i_i W_{xx} W + j_i U_{xxx} W + j_i W_{xxx} W + j_i W_{xx} W + j_i U_{xx} W + j_i W_{xx} W = 0 \]  

(12)

Some terms above will vanish automatically since they satisfy (10), then, the only terms left are,

\[ a_i W_{xxx} + b_i W_{xx} W_x + b_i W_{xx} W + b_i W_{xxx} W + b_i W_{xx} W + c_i W_{xxx} W + c_i W_{xx} W + d_i W_{xxx} W + d_i W_{xx} W + e_i W_{xxx} W + e_i W_{xx} W + e_i W_{xxx} W + e_i W_{xx} W + f_i W_{xxx} W + f_i W_{xx} W + g_i W_{xxx} W + g_i W_{xx} W + h_i W_{xxx} W + h_i W_{xx} W + i_i W_{xxx} W + i_i W_{xx} W + j_i W_{xxx} W + j_i W_{xx} W = 0 \]  

(13)

\[ \text{Int. J. of Appl. Math and Mech. 10 (2): 1-20, 2014.} \]
It is interesting to note that more general solutions to (10) can be found by substituting additional terms which then resemble the following,

\[ P = U + W + \ldots \quad (14) \]

Therefore, according to the solution of continuity, a full solution in terms of the potential function is,

\[ \Phi = \{ U(x,y,\xi) + W(x,y,\xi) + \ldots \} \left[ e^{-\int \left( \sum_c [C_c e^{iC_c}] \right) dy} \right] d\xi \quad (15) \]

By implementing the coordinate relation (2c) the analytical solution is obtained.

Now the resulting velocity vectors can be produced by applying equation (2a). Substituting the velocity vectors into the Navier Stokes equations in (1a) to obtain the pressure. This completes the proof of theorem 1.

However, the pressure relation can also be applied to the modified Navier-Stokes equations

\[ \frac{1}{\rho} \nabla^2 p = -\nabla V \cdot \nabla V \quad (16) \]

The other terms are dropped by the continuity equation. It can easily be noticed that by substituting the known expressions of the previous result for velocity, equation (16) becomes a linear partial differential equation and the pressure relation is also solved by this procedure.

### 3 IMPLEMENTATION OF THE METHOD

One of the crucial problems in the theory of differential equations is finding and studying classes of important equations that are integrable in closed form and, in particular, possess explicit solutions. It is known that a particular class of the solution of nonlinear differential equations can be obtained by several procedures (Abdel-Gawad, 1999, Seng Zhang, 2008, Musslimani et al., 2008). In this section, a new method is also proposed in order to solve nonlinear differential equations. Introducing \( Q = P_x \), equation (10) will then transform to,

\[\begin{align*}
&\left[ a_1 \frac{\partial Q}{\partial P} \right]^2 + a_2 \frac{\partial Q}{\partial P} \frac{\partial^2 Q}{\partial P^2} + a_3 Q \frac{\partial^2 Q}{\partial P^2} + b_1 Q \left( \frac{\partial Q}{\partial P} \right)^2 + b_2 Q \frac{\partial^2 Q}{\partial P^2} + c_1 \frac{\partial Q}{\partial P} \frac{\partial^2 Q}{\partial P^2} + d_1 Q \left( \frac{\partial Q}{\partial P} \right)^2 + d_2 Q \frac{\partial^2 Q}{\partial P^2} + f_1 P \frac{\partial Q}{\partial P} + g_1 Q \frac{\partial Q}{\partial P} + h_1 Q + j_1 Q = 0
\end{align*}\]

Considering the following polynomial \( Q \) and substitute into (17),

\[ Q = P_x = a_1 P^2 + b_2 P + c_1 \quad (18a) \]

If the coefficients in (18a) are taken as arbitrary values, then the following system is produced from (17),

\[\ldots\]
\[ Q = P_z = a_5 P^2 + b_5 P + c_5 \tag{18b} \]

For more general solution, equation (11) is substituted into (10) and will generate the relation below,

\[ a_3 W_{xxxx} + b_3 W_{xxx} W_x + c_3 W_{xx} W + k_1(x) W_{xxx} + e_3 W_{xx} W_x + f_3 W_{x} W + k_2(x) W_{xx} + h_3 W^2 + i_3 W_x W + k_3(x) W_x = 0 \tag{19a} \]

where \( k_1(x), k_2(x) \) and \( k_3(x) \) are clearly dependent on \( U \). By applying the same method, the corresponding equation is then,

\[ W_x = m_1(x) W^2 + m_2(x) W + m_2(x) \tag{19b} \]

\[ m_4(x) W^4 + m_5(x) W^3 + m_6(x) W^2 + m_7(x) W + m_8(x) = 0 \tag{19c} \]

In order to solve the system of (19b), the following step is necessary,

Lemma 3: Consider equation (19b) and set \( m_2 = f_1 - \frac{f_2 x}{f_2} \) to generate,

\[ Z_x = \frac{m_1}{C_6 f_2} Z^2 + f_1 Z + C_6 f_2 m_3 \]

where \( Z = C_1 f_2 W \). There exists a function \( \alpha \) such that \( f_2 m_3 = \alpha Z \), which generate the Bernoulli equation. The Riccati equation then has a closed-form exact solution when \( \alpha \) is solvable.

Proof: Set \( m_3 = f_1 - \frac{f_2 x}{f_2} \) to rearrange equation (19b) as,

\[ W_x + \frac{f_2}{f_2} W = \frac{1}{C_6 f_2} \left( C_6 f_2 W \right)_x = m_1 W^2 + f_1 W + m_3 \tag{20a} \]

where \( C_6 \) is a constant. Suppose that \( Z = C_6 f_2 W \), then the following equation is produced,

\[ Z_x = \frac{m_1}{C_6 f_2} Z^2 + f_1 Z + C_6 f_2 m_3 \tag{20b} \]

Set \( f_2 m_3 = \alpha Z \), the original problem is transformed into,

\[ Z_x = \frac{m_1}{C_6 f_2} Z^2 + \left( f_1 + C_6 \alpha \right) Z \tag{20c} \]

The solution of (20c) is expressed as,
\[
Z = \frac{C_6 f \chi_{3}}{\alpha} = -e^{\int \left( f + C_6 \alpha \right) dx} \int_{\chi_5}^{\chi_6} \frac{m_1}{C_6 f_2} e^{\int \left( f + C_6 \alpha \right) dx} dx
\]  
(20d)

Let \( \int_{\chi_5}^{\chi_6} \frac{m_1}{C_6 f_2} e^{\int \left( f + C_6 \alpha \right) dx} dx = D \), then \( e^{\int \left( f + C_6 \alpha \right) dx} = \frac{C_6 f_2}{m_1} D \). Solving for \( D \),
\[
D = \int_{\xi}^{\chi_6} \frac{m_1}{C_6 f_2} e^{\int \left( f + C_6 \alpha \right) d\xi} d\xi = e^{-\int \frac{m_1}{C_6 f_2} d\xi} \]  
(20e)

Set \( \alpha = \frac{A}{B} \) and differentiate (20e) once to give,
\[
\frac{A}{C_6 f_2} e^{\int \left( f + C_6 \alpha \right) d\xi} = -m_3 B e^{-\int \frac{m_1}{C_6 f_2} d\xi} = g
\]  
(20f)

Thus, \( A \) and \( B \) are given by,
\[
A = \frac{f_2 g e^{\int \frac{f d\xi}{C_6}}} {C_6 \int_{\xi}^{\chi_6} \frac{f_2 g e^{\int \frac{f d\xi}{C_6}}} {C_6} d\xi} \quad \text{and} \quad B = -\frac{g}{m_3 \int_{\xi}^{\chi_6} \frac{1}{C_6} g m d\xi}
\]  
(20g)

The function \( \alpha \) can be determined as,
\[
\alpha = \frac{A}{B} = \frac{f_2 m_3 e^{\int \frac{f d\xi}{C_6}}} {C_6 \int_{\xi}^{\chi_6} \frac{1}{C_6} g m d\xi}
\]  
(20h)

Without loss of generality suppose that, \( f_2 g e^{\int \frac{f d\xi}{C_6}} = A \Phi \) to produce,
\[
\frac{A}{B} = -\frac{f_2 m_3 e^{\int \frac{f d\xi}{C_6}}} {C_6 \int_{\xi}^{\chi_6} \frac{m_1}{C_6} e^{\int \frac{f d\xi}{C_6}} d\xi}
\]  
(20i)

The solution for \( \alpha \) is then,
\[
\alpha = \frac{A}{B} = \frac{C_7}{\Phi} \left[ \int_{\xi}^{\chi_6} \left( f_2 \Phi m_3 e^{\int \frac{f d\xi}{C_6}} \int_{\xi}^{\chi_6} \frac{m_1}{C_6} e^{\int \frac{f d\xi}{C_6}} d\xi \right) d\xi \right]^{\frac{1}{2}}
\]  
(20j)

The above equation is combined with (20d) to form the solution of the general Riccati equation. This proves lemma 3.
Thus, the system represented by (19b) and (19c) will have a simultaneous solution as it is driven by lemma 3 and the polynomial equation. The claim is concluded in the following theorem.

Theorem 2: Consider the solution of the general Riccati equation as described by (20d) and (20j). By combining with the root of polynomial equation, $W = \beta(\xi)$, then the expressions of $f_1$ and $f_2$ can be determined. The resulting expressions thus complete the solution of the system defined by (19b) and (19c).

Proof of theorem 2:

Equating the results from (19c) and equation (20) as follows,

$$\frac{m_3}{\alpha} = \beta \text{ or } m_3 = \beta \alpha = \frac{C_7}{\Phi} \left[ \int_{\xi} f_2 \Phi m_3 e^{-\int f_d \xi} d\xi \right]_{\xi}$$  \hspace{1cm} (21a)

Integrate the above equation and perform the algebraic calculations to give,

$$\int_{\xi} m_3 \Phi e^{\int_{\xi} f_d \xi} d\xi = C_8 \left( \int_{\xi} \beta \Phi d\xi \right)^2 = \frac{\varphi}{f_2} e^{\int_{\xi} f_d \xi}$$  \hspace{1cm} (21b)

with $\varphi = \frac{C_8}{m_3 \Phi} \left( \int_{\xi} \beta \Phi d\xi \right)^2$. Differentiate (21b) once to get the relations of $f_1$ and $f_2$ as in the following,

$$\left( \frac{1}{f_2} \right)_{\xi} = \frac{1}{f_2} \left( \frac{\Phi m_3}{\varphi} \frac{\varphi}{f_1} \right)$$  \hspace{1cm} (21c)

The solution for $f_2$ is then,

$$f_2 = \varphi e^{\int_{\xi} f_d \xi}$$  \hspace{1cm} (21d)

Equating the above equation with the solution of $m_2 = f_1 - \frac{f_2}{f_2}$ to get the following expression,

$$m_2 = f_1 + \left( \frac{\Phi m_3}{\varphi} \frac{\varphi}{\varphi} - f_1 \right)$$  \hspace{1cm} (21e)

It is important to mention that the solution for $f_1$ does not exist. This condition is consistent with equation (21d), since $f_1$ will vanish when (21d) is substituted into (20j).
\[
W = \frac{m_b}{\alpha} = -e^{\int (f_1 + C_\alpha) d\xi} = -e^{\int \frac{\Phi}{\varphi} (f_1 + C_\alpha) d\xi}
\]

with, \( \Phi = \frac{\beta}{m_3} \left[ \frac{2C_h}{\beta} \right] \left[ \frac{2m_3 C_h}{C_h} \right] \exp \left[ \int \xi \left[ \frac{2C_h}{\beta} \right] \left[ \frac{2m_3 C_h}{C_h} \right] d\xi \right] \) and the function \( \alpha \) are given by

\[
\alpha = \frac{A}{B} \left[ \int \xi \left( f_2 \Phi m_3 e^{-\int f_1 d\xi} \int \xi \left( m_1 \Phi e^{\int f_1 d\xi} d\xi \right) \right) \right]^2 \]

(Nugroho, 2013). This completes the proof of theorem 2.

Lemma 3 is also applied for (18b), and the solution of \( P \) is obtained in a functional series. It is interesting to note that higher order polynomial equations can also be produced by the proposed procedure through factoring their polynomials and integrating their terms as the keystones.

4 PROPERTIES OF THE SOLUTIONS

It is interesting to analyze the contribution of the solution to the energy rate in the flow field, which is stated in the following proposition,

Proposition: If the rate of energy in the whole domain of \( \Xi, i = 1, ..., m \) and \( \Xi_{ai}, i = 1, ..., n \) with respect to the initial and boundary conditions \( V = V(x, y, z, 0) \) on \( \partial \Xi_1 \) or \( \partial \Xi_{ai} \), is equal to zero, then \( p \) and \( V \) as solutions to the continuity and the incompressible Navier-Stokes equations will be constant in \( \Xi \) and \( \Xi_{ai} \).

Proof: Let \( \Xi \) be the region of interest. It is supposed that the associated problem is a connected, bounded region in three-dimensional domain. Moreover, let \( \Xi, i = 1, ..., m \), be sub regions characterized by simple boundaries and \( \Xi_{ai}, i = 1, ..., n \) be the sub regions where boundaries are not simple. This means that the considered boundaries \( \partial \Xi_1 \) and \( \partial \Xi_{ai} \) are defined as regular and irregular surfaces, respectively.

Thus, the boundary value problem is determined as follows; given density \( \rho > 0 \) such that velocity \( V \) is a real vector field consists of \((u, v, w)\) components and \( p \) is a real scalar field defined in every region \( \Xi, i = 1, ..., m \), and \( \Xi_{ai}, i = 1, ..., n \) which fulfill the incompressible Navier-Stokes equations and continuity in (1a) and (1b) with boundary and initial conditions \( V = V(x, y, z, 0) \) on \( \partial \Xi_1 \) or \( \partial \Xi_{ai} \). The kinematic viscosity \( \nu > 0 \), \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \) and \( \vec{n} \) is the unit vector normal to the surfaces \( S \) parallel to the velocity.

The energy rate is defined as a product of static pressure \( p \) and flow rate \( V \) across control surface. Direction of velocity is parallel to the unit normal control surface \( \vec{n} \). It is supposed
the region $\Xi$ be the associated problem and it consists of $i$ parts with simple and non simple boundaries. Therefore, the assumed zero rate energy can be written as,

$$\frac{\partial E}{\partial t} = pAV + \frac{1}{2}pV^2AV = 0$$

(22a)

Consequently, the divergence theorem can be applied to the whole region of interest as,

$$\oint_S \left( pV\n + \frac{1}{2}pV^2V\n \right)dS = \int_\Xi \nabla \cdot \left( pV + \frac{1}{2}pV^2 \right)d\Xi = 0$$

(22b)

By using vector identity, $\nabla \cdot (fF) = f\nabla \cdot F + F \cdot \nabla f$, hence, it is identified,

$$\oint_S \left( pV\n + \frac{1}{2}pV^2V\n \right)dS = \int_V \left( p\nabla V + V\nabla p + V^2\nabla V + V\nabla V^2 \right)dV$$

(22c)

Since $\oint_S \left( pV\n + \frac{1}{2}pV^2V\n \right)dS = 0$, then,

$$\sum_{i=1}^{m} \int_{\Xi_i} \left( p\nabla V + V\nabla p + V^2\nabla V + V\nabla V^2 \right)d\Xi + \sum_{i=1}^{n} \int_{\Xi_n} \left( p\nabla V + V\nabla p + V^2\nabla V + V\nabla V^2 \right)d\Xi = 0$$

(22d)

Suppose that the rate of energy is zero everywhere, then, equation (22d) is automatically satisfied. The first and third terms of energy rate are always zero according to continuity. Therefore, since $p$ and $V$ are nonzero, then $\nabla p$ and $\nabla V$ must be zero. Thus, the following trivial solution is defined,

$$p = \text{constant and } V = \text{constant}$$

(22e)

It is noted that (22e) will also satisfy the continuity and incompressible Navier-Stokes equations. This proves the proposition.

The proposition indicates the impossibility of nontrivial solution to have zero rate energy. Suppose that energy is produced if the solution posses non triviality condition as in (21f), it is plausible that the excess energy will be distributed to the whole domain, then the observed parameters will also deviate from trivial conditions. This feature will support the qualitative aspect of turbulent flows and this represents the common properties of turbulent flows. Moreover, turbulence can be generated by implementing boundary conditions in direct numerical simulation. This implies that turbulence is also a boundary value problem of the Navier-Stokes equations as also stated by Adomian (1994). Furthermore, the case considered here strictly admits continuity in the form of zero velocity divergence. For compressible flow cases where velocity divergence is not zero, triviality in $p$ and $V$ may not exist.

Corollary: Any non trivial solution of initial boundary value problems of the continuity and the incompressible Navier-Stokes equations has non zero energy rate.
5 SOLUTION EXAMPLES AND DISCUSSIONS

In this section, the coefficients of Riccati equation will be determined to produce the known solutions. The example are conducted by the method of exact integral evaluation which is implemented in order to maintain the accuracy of the analytical solutions (Nugroho, 2012). The collections of analytical solutions to the simplified Navier-Stokes equations can be investigated in the literatures (Nugroho, 2010). Many solutions of the Navier-Stokes equations are unstable and can hardly have physical meanings. However, the method which is proposed in this work may handle physically reasonable solutions.

The above procedure shows that the physically simple and useful solutions can be produced within a specific initial-value if the coefficients are carefully chosen. However, the construction of analytical solution in this work may lead to the sensitivity with respect to initial condition because the method is usually performed forward. Moreover, it is found that $u$ is forced in order to be unique and $p$ is not unique which can yield the unstable and fluctuating solutions and leads to turbulent solutions. Therefore, it is noted that the method may also valid for turbulent flow problems. According to the proposition which was stated in the previous work that any non trivial solutions can be regarded as turbulent solution (Nugroho, 2010), one can investigate the nature of the solution to describe the structure of turbulent flows. It is well accepted that turbulence structures are represented by the condition of non zero mean fluctuation (Holmes et al., 1996). Based on the decomposition scheme, $w = \bar{w} + w'$ and (21f) it is not hard to verify that the relation $\bar{w} = \bar{w}'$ is hold such that $\bar{w} = \bar{w} + \bar{w}'$. Therefore, $\bar{w}' \neq 0$, which is also applied to $v, u$ and $p$ as well.
Figure 1. The fluctuation of velocity through time

Figure 1 shows the fluctuation velocity through time with different time samplings in the small scale. It is also shown that the velocity is decaying in the small scale, this is due to the dissipation mechanism of the turbulent flows which convert energy from large scale into heat at the molecular level.
The other aspect of the solution is depicted in figure 2, the velocity accumulates rapidly with the extended time. The it is clear from the integral evaluation that the solution become bounded if the coefficient appeared in the Riccati equation is also bounded. This striking features will be the subject of further investigation of the mathematical properties of the solutions.

Figure 2. The rapidly increasing velocity through time
Figure 3 are finally show the bifurcation phenomena which appeared in the solutions. The phenomena are detected by plotting the real and imaginary parts of the solutions, which explicitly shows the strong nonlinearity of the governing equations. This special feature is known for the incompressible Navier-Stokes equations by the qualitative analysis of dynamical systems which deserves further investigations in explicit solutions.

6 CONCLUSION

The analytical solutions of the three-dimensional incompressible flows are introduced in this paper. The solution is proposed to be in the form $V = \nabla \Phi + \nabla \times \Phi$ where $\Phi$ is a potential function that takes the form $\Phi = P(x,y,\zeta)R(y)S(\zeta)$. The potential function is decomposed into a
product of functionals $P$, $R$, and $S$ in order to reduce the whole problems into one of solving a nonlinear ordinary differential equation in $P$ and linear differential equations in $R$ and $S$. Then, the explicit solutions for $R$ and $S$ are obtained through the continuity equation. The potential function is then substituted into the vorticity equations to reduce it to a class of nonlinear ordinary differential equation in term of $P$. The general solution for $P$ is derived based on the known particular solution. Then, the pressure is solved by applying the velocity vector into the Navier-Stokes equations to complete the solutions. The uniqueness property is proved by generating and equating two solutions. For the case of blow up, the solution is regularized by introducing different function which converges to the original function a controllable error. The role of simple energy equation is also considered by a standard method of vector identity. The result with respect to initial and boundary conditions shows that the energy rate is not zero for any nontrivial solution which also represents a qualitative property of turbulence. The results show the fluctuation of the decaying velocity through time due to energy dissipation. It also reveals the rapid velocity accumulations providing that the solutions may produce singularity in the small scale of turbulent flows. The bifurcation is then detected which shows the strong nonlinearity in the small scale of turbulent flows.

Regarding the coordinate transformation, selection of variables in the potential function can be interchanged from the beginning. Instead of using the coordinate relation (2c) and potential function (4) the following expression can be used,

$$\Phi = P(\xi, y, z) R(y) S(z), \quad \xi = kx - \zeta(t) \text{ or } \Phi = P(x, \xi, z) R(\xi) S(z), \quad \xi = ky - \zeta(t)$$

(23)

In particular, it is reasonable that other classes of nontrivial analytical solutions may still be developed from (10) or from the original Navier-Stokes equations by more complex procedures to know more about properties of the analytical solutions (Popovich, 1995). Then applying initial and boundary conditions, the problem of uniqueness can be settled. It is important to note that the convergence of (14) have to be ensured to make the solution meaningful. Furthermore, the generality of $\zeta(t)$ will make the obtained solutions open for further investigations in the problem of regularizations.

REFERENCES


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