A NEW APPLICATION OF USING HOMOTOPY ANALYSIS METHOD FOR SOLVING STOCHASTIC QUADRATIC NONLINEAR DIFFUSION EQUATION

M. A. El-Tawil\textsuperscript{1}, H. N. Hassan\textsuperscript{2}, and A. F. Fareed\textsuperscript{3}

\textsuperscript{1}Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University, Egypt.
\textit{Email: magdyeltawil@yahoo.com}

\textsuperscript{2,3}Department of Basic Science, Faculty of Engineering at Benha, Benha University, Benha, 13512, Egypt.
\textit{Email: h_nasr77@yahoo.com}
\textit{Email: aisha_farid@yahoo.com}

Received 30 June 2012; accepted 29 July 2013

ABSTRACT

The homotopy analysis method (HAM) is applied to obtain some approximate orders of mean and variance for the solution process quadratic nonlinear diffusion equation under stochastic non homogeneity. The scheme shows importance of choice of convergence control parameter \( h \) to guarantee the convergence of the solution. The method provides the solution in the form of a rapidly convergent series with easily computable components using symbolic computation software such as Mathematica. The method of solution is illustrated through figures, comparisons among different methods and some parametric studies have been done to show accuracy of the presented method.

Keywords: Homotopy analysis method, Stochastic nonlinear differential equations, Diffusion equation

1 INTRODUCTION

The study of random solutions of partial differential equations was initiated by (Kampe 1955). In his valuable survey on the theory of random equations, Bharucha-Reid showed how a stochastic heat equation of Cauchy type can be solved using the stochastic integrals theory (Bharucha 1965). (Lo Dato 1973), considered the stochastic velocity field and the Navier-Stokes equation and discussed the mathematical problems associated with it. (Becus 1977) introduced a general solution for the heat conduction problem with a random source term and random initial and boundary conditions. Many authors investigated the stochastic diffusion equation under different views, see (Manthey 1986; Jetschke 1986; El-Tawil 1996; Uemura 1996). The homotopy analysis
The homotopy analysis method (HAM) is an analytical technique for solving nonlinear differential equations. Initially proposed by Liao in his Ph.D. thesis (Liao 1992) the technique is superior to the traditional perturbation methods in that it leads to convergent series solutions of strongly nonlinear problems, independent of any small or large physical parameter associated with the problem, (Liao 1992). The HAM provides a more viable alternative to nonperturbation techniques such as the Adomian decomposition method (ADM) (Adomian 1991; Rach 1984) and other techniques that cannot guarantee the convergence of the solution series and may be only valid for weakly nonlinear problems, (Liao 2009). We note here that He’s homotopy perturbation method (HPM), (He 2003) is only a special case of the HAM (Liao 2005). Indeed Liao (Liao 2003) makes a compelling case that the Adomian decomposition method, the Lyapunov artificial small parameter method and the-expansion method are nothing but special cases of the HAM. In recent years; this method has been successfully employed to solve many in science and engineering such as the viscous flows of non-Newtonian fluids (Hayat, Khan, Sajid and Asghar 2007; Hayat and Sajid 2007; Sajid, Hayat and Asghar 2007), the KdV-type equations (Abbasbandy 2008; Liu and Li ZB 2009; Song and Zhang 2007), Glaucert-jet problem (Bouremel 2007), Burgers–Huxley equation (Molabahrami and Khani 2009), time-dependent Emden–Fowler type equations (Bataineh, Noorani and Hashim 2007), differential-difference equation (Wang, Zou and Zhang 2007), the MHD Falkner-Skan flow (Abbasbandy and Hayat 2009), multiple solutions of nonlinear problems (Xu and Liao 2008; Abbasbandy and Shivanian E 2010; Abbasbandy, Magyari, Shivanian 2009), Schrödinger equations (Hassan and El-Tawil 2011c), two-point nonlinear boundary value problems (Hassan and El-Tawil 2011b), a new technique of using homotopy analysis method for solving high-order nonlinear differential equations (Hassan and El-Tawil 2011 a), and more other applications (Abbasbandy, Ashtiani and Babolian 2010; Tan and Abbasbandy S 2008; Rashidi, Hayat, Erfani, Mohimanian and Awatif 2011), Modified Homotopy Analysis Method for Second order nonlinear initial value problems (Hassan and El-Tawil 2011 d).

The homotopy analysis method applied for finding the mean and variance of the stochastic quadratic nonlinear equation with $\sigma n(x;\omega)$ as non homogeneity given by (El-Tawil and El-Mulla 2010)

$$\frac{\partial u(t,x;\omega)}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \epsilon u^2 + \sigma n(x;\omega); \ (t,x) \in (0,\infty) \times (0, l),$$

$$u(t,0;\omega)=0, u(t,l;\omega)=0 \text{ and } u(0,x;\omega)=\phi(x).$$

Where $u(t,x;\omega)$ is the diffusion process, $\epsilon$ is a deterministic scale for the nonlinear term. $\omega$ is an outcome of a triple probability space $(\Omega, \chi, P)$ in which $\Omega$ is a sample space, $\chi$ is $\sigma$- field associated with $\Omega$ and $P$ is a probability measure The in homogeneity term $\sigma n(x;\omega)$ is space white noise scaled by $\sigma$ which has the following important properties:
\[ E \ n(x; \omega) = 0 \]
\[ E \ n(x_1; \omega)n(x_2; \omega) = \delta(x_1 - x_2) \quad (2) \]

Where \( E \) denotes the ensemble average operator, \( \delta(\cdot) \) is the Dirac delta function.

2 THE BASIC IDEA OF HAM

A presentation of the standard HAM for deterministic problems can be found in (Liao 1992; Liao 2009). The following subsection is a brief description of HAM.

To describe the basic ideas of HAM, we consider the following differential equation:
\[ N[u(t, x)] = 0 \quad (3) \]

Where \( N \) is a nonlinear operator, \( x \) and \( t \) denote independent variables, and \( u(t, x) \) is an unknown function. By means of generalizing the traditional homotopy method, Liao (Liao 1992; 2003) constructs the so-called zero-order deformation equation
\[ (1 - q)L[(\phi(t, x; q) - u_0(t, x))] = qhH(t, x)N[\phi(t, x; q)], \quad (4) \]

Where \( q \in [0, 1] \) denote the so-called embedding parameter, \( h \neq 0 \) is an auxiliary parameter and \( L \) is an auxiliary linear operator.

The HAM is based on a kind of continuous mapping \( u(t, x) \rightarrow \phi(t, x; q), \phi(t, x; q) \) is an unknown function, \( u_0(t, x) \) is an initial guess of \( u(t, x) \), and \( H(t, x) \) denotes a non-zero auxiliary function. It is obvious that when the embedding parameter \( q = 0 \) and \( q = 1 \), Equation (3) becomes
\[ (1 - q)L[(\phi(t, x; q) - u_0(t, x))] = qhH(t, x)N[\phi(t, x; q)], \quad (5) \]

Respectively. Thus as \( q \) increases from 0 to 1, the solution \( \phi(t, x; q) \) varies from the initial guess \( u_0(t, x) \) to the solution \( u(t, x) \). In topology, this kind of variation is the called deformation, Equation (4) construct the homotopy \( \phi(t, x; q) \), and (4) is called the zero-order deformation equation.

Having the freedom to choose the auxiliary parameter \( h \), the auxiliary function \( H(t, x) \), the initial approximation \( u_0(t, x) \), and the auxiliary linear operator \( L \), we can assume that all of them are properly chosen so that the solution \( \phi(t, x; q) \) of the zero-order deformation Equation (4) exists for \( 0 \leq q \leq 1 \).

Expanding \( \phi(t, x; q) \) in the Taylor series with respect to \( q \), one has
\[ \phi(t, x; q) = u_0(t, x) + \sum_{m=1}^{\infty} u_m(t, x) q^m, \]  \hspace{1cm} (6)

Where
\[ u_m(t, x) = \frac{1}{m!} \left. \frac{\partial^m \phi(t, x; q)}{\partial q^m} \right|_{q=0} \]  \hspace{1cm} (7)

Assume that the auxiliary parameter \( h \), the auxiliary function \( H(t, x) \), the initial approximation \( u_0(t, x) \) and the auxiliary linear operator L are so properly chosen that the series (6) converges at \( q = 1 \) and
\[ \phi(t, x; 1) = u_0(t, x) + \sum_{m=1}^{\infty} u_m(t, x), \]  \hspace{1cm} (8)

Which must be one of the solutions of the original nonlinear equation, as proved by Liao (Liao 2009) As \( h = -1 \), and \( H(t, x) = 1 \), (4) becomes
\[ (1 - q) L [(\phi(t, x; q) - u_0(t, x)] + q N[\phi(t, x; q)] = 0, \]  \hspace{1cm} (9)

This is mostly used in the homotopy-perturbation method (Sajid, Hayat and Asghar 2007).

According to definition (8), the governing equation and the corresponding initial condition of \( u_m(t, x) \) can be deduced from the zero-order deformation equation (4). Define the vector
\[ u_n(t, x) = \{ u_0(t, x), u_1(t, x), u_2(t, x), \ldots, u_n(t, x) \}. \]

Differentiating Equation (4) \( m \) times with respect to the embedding parameter \( q \) and then setting \( q = 0 \) and finally dividing them by \( m! \), we have the so-called \( m\text{-th} \)-order deformation equation:
\[ L[u_m(t, x) - \chi_m u_{m-1}(t, x)] = h H(t, x) R(u_{m-1}), \]  \hspace{1cm} (10)

Where
\[ R(u_{m-1}) = \frac{1}{m-1!} \left. \frac{\partial^{m-1} N[\phi(t, x; q)]}{\partial q^{m-1}} \right|_{q=0}, \]  \hspace{1cm} (11)

And
\[ \chi_m = \begin{cases} 0 & \text{when } m \leq 1, \\ 1 & \text{otherwise...}. \end{cases} \]  \hspace{1cm} (12)
\( u(t, x) = \sum_{i=0}^{\infty} u_i(t, x) \). \hspace{1cm} (13)

It should be emphasized that \( u_m(t, x) \) for \( m \leq 1 \) is governed by the linear equation (10) with linear boundary conditions that come from the original problem, which can be solved by the symbolic computation software such as Mathematica, Maple, and Matlab.

3 APPLICATION OF HAM ON STOCHASTIC QUADRATIC NONLINEAR DIFFUSION EQUATION

To demonstrate the performance of the presented method it will be used to find mean and variance of stochastic quadratic nonlinear diffusion problem like follows.

The auxiliary linear operator chosen as

\[
L[\phi(t, x; q)] = \frac{\partial^2 \phi(t, x; q)}{\partial t^2} - \frac{\partial^2 \phi(t, x; q)}{\partial x^2} \hspace{1cm} (14)
\]

We have many choices in guessing the initial approximation together with its initial conditions which greatly affects the consequent approximation. The choice \( u_0 \) is a design problem which can be taken as follows:

\[
u_0(t, x) = \sum_{n=0}^{\infty} B_n e^{i\beta n t} \sin \frac{n\pi x}{L} \hspace{1cm} (15)
\]

\[
B_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} \, dx
\]

One can notice that the selected value function satisfies the initial and boundary conditions and it depends on the parameter \( \beta_n \) which is totally free. One can also notice that \( \beta_n \) selection could control the solution convergence.

Furthermore, we define the nonlinear operator as

\[
N[\phi(t, x; q)] = \frac{\partial \phi(t, x; q)}{\partial t} - \frac{\partial^2 \phi(t, x; q)}{\partial x^2} + \varepsilon \cdot [\phi(t, x; q)]^2 - \sigma n(x; \omega); \hspace{1cm} (16)
\]

We construct the so-called zero-order deformation equation,

\[
(1 - q) L[u_m(t, x) - \chi_m u_{m-1}(t, x)] = q h H(t, x) R(u_{m-1}). \hspace{1cm} (17)
\]

The \( m \)th-order deformation equation for \( m \geq 1 \) and \( H(t, x) = 1 \) is.
\[ L[(u_m(t,x) - \chi_m u_{m-1}(t,x)] = hR(\tilde{u}_{m-1}), \]  

Subject to boundary conditions

\[ u_m(t,0)=0, u_m(t,l)=0, \]  

And initial condition

\[ u_m(0,x)=0, \]

Where \( \chi_m \) is defined by (12) and

\[ R(\tilde{u}_{m-1}) = \frac{\partial u_{m-1}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}(t,x)}{\partial x^2} + \varepsilon \sum_{i=0}^{m-1} u_{m-1-i}(t,x)u_i(t,x) - \sigma n(x;\omega); \]

Now the solution of the \( m \)-th order deformation equation (18) for \( m \geq 1 \) become

\[ L[(u_m(t,x) - \chi_m u_{m-1}(t,x)] = h\left[ \frac{\partial u_{m-1}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}(t,x)}{\partial x^2} + \varepsilon \sum_{i=0}^{m-1} u_{m-1-i}(t,x)u_i(t,x) - \sigma n(x;\omega) \right]. \]

The first order approximation is obtained by substituting with \( m=1 \) in (18) as follows

\[ L[u_1(t,x)] = hR(\tilde{u}_0) \]

Where

\[ R(\tilde{u}_0) = \frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2 -\sigma n(x;\omega); \]

Then

\[ L[u_1(t,x)] = h\left[ \frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2 -\sigma n(x;\omega) \right], \]

The approximated first order solution of (25) can be obtained using Eigen function expansion as follows,
\[ u_i(t,x) = \sum_{n=0}^{\infty} I_{n,1}(t) \sin \frac{n\pi}{L} x \]

where

\[ I_{n,1}(t) = \int_0^t e^{-\frac{n\pi^2}{L^2} (t-\tau)} F_{n,1}(\tau) d\tau \]

\[ F_{n,1}(t) = \frac{2h}{L} \int_0^L \left[ \frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2 - \sigma n(x,\omega) \right] \sin \frac{n\pi}{L} x dx, \]  

(26)

The ensemble average of the first order approximation is

\[ \mu[u_i(t,x)] = \sum_{n=0}^{\infty} E(I_{n,1}(t)) \sin \frac{n\pi}{\ell} x \]

where

\[ E(I_{n,1}(t)) = \int_0^t e^{-\frac{n\pi}{\ell} (t-\tau)} E(F_{n,1}(\tau)) d\tau \]

\[ E(F_{n,1}(t)) = \frac{2h}{\ell} \int_0^L \left[ \frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2 \right] \sin \frac{n\pi}{\ell} x dx \]

\[ \mu[u_i(t,x)] = \frac{e^{-\pi^2 h(3(1 + e^{(\pi^2 + 2\beta n)\pi})\pi(\pi^2 + 2\beta n) - 8\varepsilon + 8e^{\pi(\pi^2 + 2\beta n)\pi})\sin[\pi x]} \pi^2}{3(\pi^2 + 2\pi\beta n)} \]  

(27)

The covariance of the first order solution can have the following expression

\[ \text{Cov}[u_i(t,x_1),u_i(t,x_2)] = E[(u_i(t,x_1) - Eu_i(t,x_1))(u_i(t,x_2) - Eu_i(t,x_2))] \]

\[ = E[\left( \sum_{n=1}^{\infty} (I_{n,1}(t) - EI_{n,1}(t)) \sin \frac{n\pi}{L} x_1 \right) \left( \sum_{m=1}^{\infty} (I_{m,1}(t) - EI_{m,1}(t)) \sin \frac{m\pi}{L} x_2 \right)] \]  

(28)

Where Cov denotes the covariance operator.

The covariance is obtained from the following final expression

\[ \text{Cov}(u_i(t,x_1),u_i(t,x_2)) = \frac{4h^2 \sigma^2}{\ell^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{n\pi}{\ell} x_1 \sin \frac{m\pi}{\ell} x_2 \left( \int_0^t \int_0^t e^{-\frac{n\pi}{\ell} (t - \tau_1)} e^{-\frac{m\pi}{\ell} (t - \tau_2)} d\tau_1 d\tau_2 \right) \]

\[ \text{Cov}(u_i(t,x_1),u_i(t,x_2)) = \frac{2(1 - e^{-\pi^2 h^2} \sin[\pi x_1] \sin[\pi x_2])}{\pi^4} \]  

(29)
The variance of the first order solution can have the following expression

\[
Var[u_t(t, x)] = E[u_t(t, x) - Eu_t(t, x)]^2 = E[(\sum_{n=1}^{\infty} (I_{\eta_1}(t) - EI_{\eta_0}(t)) \sin \frac{n\pi}{L} x)]^2
\]

(30)

Where Var denotes the variance operator.

The variance can then be obtained from equation (29) by setting \( x_1 = x_2 = x \)

\[
Var[u_t(t, x)] = \frac{4h^2 \sigma^2}{\ell^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{n\pi}{\ell} x \sin \frac{m\pi}{\ell} x (\int_0^t \sin \frac{n\pi}{\ell} x \sin \frac{m\pi}{\ell} x dx)
\]

\[
\left( \int_0^t \int_0^t e^{-\frac{(\eta_1 - \tau_1)^2}{\ell^2}} e^{-\frac{(\eta_2 - \tau_2)^2}{\ell^2}} d\tau_1 d\tau_2 \right)
\]

\[
Var[u_t(t, x)] = \frac{2(1 - e^{-\pi^2 \tau})^2 h^2 \sin[\pi x]^2}{\pi^4}
\]

(31)

The second order approximation is obtained by substituting with \( m=2 \) in (18) as follows

\[
L[u_2(t, x) - u_t(t, x)] = hR(\bar{u}_t)
\]

(32)

Where

\[
R(\bar{u}_t) = \frac{\partial u_t(t, x)}{\partial t} - \frac{\partial^2 u_t(t, x)}{\partial x^2} + \varepsilon \sum_{i=0}^{1} u_{\eta_i}(t, x) u_t(t, x)
\]

\[
= \frac{\partial u_t(t, x)}{\partial t} - \frac{\partial^2 u_t(t, x)}{\partial x^2} + 2\varepsilon u_0(t, x) u_t(t, x),
\]

(33)

Substituting by (33) in (32) we get

\[
L[u_2(t, x) - u_t(t, x)] = h\left[ \frac{\partial u_t(t, x)}{\partial t} - \frac{\partial^2 u_t(t, x)}{\partial x^2} + 2\varepsilon u_0(t, x) u_t(t, x) \right],
\]

(34)

The solution of (34) can be obtained using Eigen function expansion as follows

\[
u_2(t, x) = u_t(t, x) + \sum_{n=0}^{\infty} I_{\eta_2}(t) \sin \frac{n\pi}{\ell} x
\]

where

\[
I_{\eta_2}(t) = \int_0^t \int_0^t e^{-\frac{(\eta_1 - \tau)^2}{\ell^2}} F_{\eta_2}(\tau) d\tau
\]

\[
F_{\eta_2}(t) = \frac{2h}{\ell} \int_0^t \frac{\partial u_t(t, x)}{\partial t} \frac{\partial^2 u_t(t, x)}{\partial x^2} + 2\varepsilon u_0(t, x) u_t(t, x) \sin \frac{n\pi}{\ell} x dx,
\]

(35)

The ensemble average of the second order solution can be obtained as

$$\mu[u_2(t,x)] = \mu[u_1(t,x)] + \sum_{n=0}^{\infty} E(I_{n,2}(t)) \sin \frac{n \pi}{\ell} x$$

where

$$E(I_{n,2}(t)) = \int_{0}^{t} e^{-\frac{\rho}{\ell} (t-\tau)} E(F_{n,2}^{(\tau)}) d\tau$$

(36)

Where

$$E(F_{n,2}^{(\tau)}) = \frac{2h}{\ell} \left[ \frac{\partial}{\partial t} \mu[u,(t,x)] - \frac{\partial^2}{\partial x^2} \mu[u,(t,x)] + \varepsilon \mu[u,(t,x)] \right] \sin \frac{n \pi}{\ell} x dx$$

$$e^{-\rho t} h(3(-1 + e^{(\pi^2 + \beta n)})\pi(\pi^2 + 2\beta n) - 8\varepsilon + 8e^{(\pi^2 + \beta n)}) \sin[\pi x] +$$

$$3(\pi^2 + 2\pi \beta n)$$

$$e^{-\rho t} h(-128\beta n e^2 - (\pi^2 + 3\beta n)(9\pi^2 \beta n(\pi^2 + 2\beta n) + 72\pi \beta n e - 16\varepsilon(3\pi^3 + 6\pi \beta n + 8\varepsilon))$$

$$+ e^{\beta n}(128e^{(\pi^2 + \beta n)} \beta n e^2 + (\pi^2 + 3\beta n)(9e^{\pi^2} \beta n(\pi^2 + 2\beta n) + 72e^{(\pi^2 + \beta n)} \beta n e$$

$$- 16\varepsilon(3\pi^3 + 6\pi \beta n + 8\varepsilon))) \sin[\pi x]) / (9\pi^2 \beta n(\pi^2 + 2\beta n)(\pi^2 + 3\beta n))$$

(37)

The covariance of the second order solution can have the following expression

$$\text{Cov}(u_2(t,x_1), u_2(t,x_2)) = E[(u_2(t,x_1) - E u_2(t,x_1))(u_2(t,x_2) - E u_2(t,x_2))]$$

$$= E[(\sum_{n=1}^{\infty} (I_{n,1}(t) - E I_{n,1}(t)) \sin \frac{n \pi}{L} x_1 + \sum_{n=1}^{\infty} (I_{n,2}(t) - E I_{n,2}(t)) \sin \frac{n \pi}{L} x_1)$$

$$\sum_{m=1}^{\infty} (I_{m,1}(t) - E I_{m,1}(t)) \sin \frac{m \pi}{L} x_2 + \sum_{m=1}^{\infty} (I_{m,2}(t) - E I_{m,2}(t)) \sin \frac{m \pi}{L} x_2]$$

(38)

$$\text{Cov}(u_2(t,x_1), u_2(t,x_2)) = \text{Cov}(u_1(t,x_1), u_1(t,x_2))$$

$$+ 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E((I_{n,1}(t) - E I_{n,1}(t))(I_{m,1}(t) - E I_{m,1}(t))) \sin \frac{n \pi}{L} x_1 \sin \frac{m \pi}{L} x_2$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E((I_{n,2}(t) - E I_{n,2}(t))(I_{m,2}(t) - E I_{m,2}(t))) \sin \frac{n \pi}{L} x_1 \sin \frac{m \pi}{L} x_2$$

(39)

$$\text{Cov}(u_2(t,x_1), u_2(t,x_2)) = \frac{1}{\pi} \frac{2 \pi}{\ell} h^2 (3(1 + e^{-\pi^2 t})^2 \pi^4 - 1 / (\pi^2 + \beta n)2e^{-2\pi^2 t} \pi^3 (2(1 + e^{-\pi^2 t})^2 \pi^2$$

$$+ 2(-1 + e^{-\pi^2 t}) \beta n + (-1 + e^{-\pi^2 t})(-1 + e^{(\pi^2 + \beta n)})\pi \varepsilon - (-1 + e^{(\pi^2 + \beta n)}) \pi^3 t e$$

$$- 4e^{-2\pi^2 t} (\pi(-4(-1 + e^{-\pi^2 t})^2 h^2 + (1 - e^{-\pi^2 t} + \pi^2 t)((-1 + e^{(\pi^2 + \beta n)})^2 + e^{(\pi^2 + \beta n)}(-1 + e^{-\pi^2 t}) \beta n^2) e$$

$$- 16e^{2\pi^2 t} \beta n^2 (1 - e^{-\pi^2 t} + \pi^2 t) \pi^2 \sin[\pi x_1^2] \sin[\pi x_2^2])$$

(40)
\[ \text{Cov}(u_1(t,x), u_2(t,x)) = E[(u_1(t,x) - Eu_1(t,x))(u_2(t,x) - Eu_2(t,x))] = \text{Cov}(u_1(t,x), u_1(t,x)) + \]
\[ + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E((I_{n_1}(t) - EI_{n_1}(t))(I_{m_2}(t) - EI_{m_2}(t))) \sin \frac{n\pi}{\ell} x_1 \sin \frac{m\pi}{\ell} x_2 \] 
\[ \quad \text{(41)} \]

\[ \text{Cov}(u_1(t,x), u_2(t,x)) = \frac{1}{\pi^2} 2h^2 (-1 + e^{-\pi^2})^2 \pi - 1 / (\pi^2 + \beta n)e^{-2\pi^2} (2(-1 + e^{\pi^2})^2 \pi^2
\]
\[ + 2(-1 + e^{\pi^2})^2 \beta n (-1 + e^{\pi^2})(-1 + e^{(\pi^2 + \beta n)} )\pi \epsilon - (-1 + e^{(\pi^2 + \beta n)} )\pi^3 \epsilon)) \sin[\pi x_1] \sin[\pi x_2] \] \[ \text{(42)} \]

The third order approximation is obtained by setting \( m = 3 \) in (18) as follows
\[ L[u_3(t,x) - u_2(t,x)] = \hat{h}R(\tilde{u}_2) \] \[ \text{(43)} \]

Where
\[ R(\tilde{u}_2) = \frac{\partial u_2(t,x)}{\partial t} - \frac{\partial^2 u_2(t,x)}{\partial x^2} + \epsilon \sum_{i=0}^{2} u_{2-i}(t,x)u_i(t,x) \]
\[ = \frac{\partial u_2(t,x)}{\partial t} - \frac{\partial^2 u_2(t,x)}{\partial x^2} + \epsilon u_1^2(t,x) + 2\epsilon u_0(t,x)u_2(t,x), \] \[ \text{(44)} \]

Substituting by (44) in (43) we get
\[ L[u_3(t,x) - u_2(t,x)] = \hat{h}[\frac{\partial u_2(t,x)}{\partial t} - \frac{\partial^2 u_2(t,x)}{\partial x^2} + \epsilon u_1^2(t,x) + 2\epsilon u_0(t,x)u_2(t,x)], \] \[ \text{(45)} \]

The solution of (45) can be obtained using Eigen function expansion as follows
\[ u_3(t,x) = u_2(t,x) + \sum_{n=0}^{\infty} I_{n,3}(t) \sin \frac{n\pi}{\ell} x \]

where
\[ I_{n,3}(t) = \int_{0}^{t} e^{-\frac{n\pi}{\ell} (t-\tau)} F_{n,3}(\tau) d\tau \]
\[ F_{n,3}(t) = \frac{2h}{\ell} \int_{0}^{t} \left[ \frac{\partial u_2(t,x)}{\partial t} - \frac{\partial^2 u_2(t,x)}{\partial x^2} + \epsilon u_1^2(t,x) + 2\epsilon u_0(t,x)u_2(t,x) \right] \sin \frac{n\pi}{\ell} x dx, \] \[ \text{(46)} \]

The ensemble average of the third order solution can be obtained as
\[
\mu[u_3(t,x)] = \mu[u_2(t,x)] + \sum_{n=0}^{\infty} E(I_{n,3}(t))\sin\frac{n\pi}{\ell}x
\]

where

\[
E(I_{n,3}(t)) = \int_0^t e^{-\frac{n\pi^2}{\ell^2}(t-\tau)} E(F_{n,3}(\tau))d\tau
\]

\[
E(F_{n,3}(t)) = \frac{2h}{\ell} \int_0^\ell \left( \frac{\partial}{\partial t} \mu[u_2(t,x)] - \frac{\partial^2}{\partial x^2} \mu[u_2(t,x)] + \varepsilon \mu[u_1(t,x)]^2 + \varepsilon \mu[u_0(t,x)] \mu[u_2(t,x)] \right) \sin\frac{n\pi}{\ell}x dx \quad (47)
\]

In this manner, we can have more results of \(\mu[u_m(t,x)]\) and \(Var[u_m(t,x)]\) obtained at \(m = 4, 5, \ldots\)

The final mean expression of is

\[
E[u(t,x)] = \sum_{m=0}^{M} u_m(t,x)
\]

\[
E[u(t,x)] = u_0(t,x) + E[u_1(t,x)] + E[u_2(t,x)] + E[u_3(t,x)] \quad (48)
\]

The final variance expression of is

\[
Var[u(t,x)] = Var[\sum_{i=1}^{N} u_i(t,x)]
\]

\[
= Cov[\sum_{i=1}^{N} u_i(t,x)] = \sum_{i=1}^{N} Var[u_i(t,x)] + \sum_{i\neq j} Cov[u_i(t,x), u_j(t,x)]
\]

\[
Var[u(t,x)] = Var[u_0(t,x)] + Var[u_2(t,x)] + Cov[u_1(t,x), u_2(t,x)] \quad (49)
\]

4 RESULT ANALYSES

In the following figures, results of the solution of stochastic quadratic nonlinear diffusion model using HAM technique are shown at \(\sigma = 1, \phi(x) = \sin\frac{n\pi}{\ell} x\) and \(\ell = 1\). Figures 3, 5 and 8 show the Plot of \(h\)-curves of second and third order mean approximation respectively for different values of time \(t\) and space variable \(x\) at \(\ell = 1, \sigma = 1, \phi(x) = \sin\frac{n\pi}{\ell} x\). According to these \(h\)-curves, it is easy to discover that the valid region of \(h\) is a horizontal line segments, Thus \(h = -0.96\). Figures 4, 6 and 9 show the Plot of \(h\)-curves of second and third order means approximation respectively for different \(\beta_n\) values.
Figure 1: The change of the mean of first order approximation $u_1$ at $\varepsilon = 1$, $\beta_n = -1$, $\hat{h} = -0.96$ ($\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).

Figure 2: The change of the variance of first order approximation $u_1$ at $\varepsilon = 1$, $\beta_n = -1$, $\hat{h} = -0.96$ ($\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).

Figure 3: The change of the mean of second order approximation $u_2$ with parameter $h$ at different $t, x$ values, $\varepsilon = 1$, $\beta_n = -1$, ($\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).
Figure 4: The change of the mean of second order approximation $u_2$ with parameter $\eta$ at different $\beta_n$ values, $\epsilon = 1, t = x = 1, (\ell = 1, \sigma = 1, \phi(x) = \sin \frac{n\pi}{\ell} x)$.

Figure 5: The change of the mean of third order approximation $u_3$ with parameter $\eta$ at different $t, x$ values, $\epsilon = 1, \beta_n = -1, (\ell = 1, \sigma = 1, \phi(x) = \sin \frac{n\pi}{\ell} x)$.

Figure 6: The change of the mean of third order approximation $u_3$ with parameter $\eta$ at different $\beta_n$ values, $\epsilon = 1, t = x = 1, (\ell = 1, \sigma = 1, \phi(x) = \sin \frac{n\pi}{\ell} x)$.

Figure 7: The change of the mean $u$ at $\varepsilon = 1$, $\beta_n = -1$, $\hat{h} = -96$, $\phi(x) = \sin \frac{n\pi}{\ell} x$.

Figure 8: The change of the mean $u$ with parameter $\hat{h}$ at different $t, x$ values, $\varepsilon = 1$, $\beta_n = -1$

Figure 9: The change of the mean $u$ with parameter $\hat{h}$ at different $\beta_n$ values, $\varepsilon = 1$, $t=x=.1$
Figure 10: The change of the mean $u$ with time $t$ at different $\varepsilon$ values, $x=1$, $\beta_n = -1$, $h = -0.96$

\[\ell = 1, \sigma = 1, \phi(x) = \sin \left(\frac{n \pi}{\ell} x \right).\]

Figure 11: The change of the mean $u$ with space variable $x$ at $t=0.1$, $\beta_n = -1$, $h = -0.96$

\[\ell = 1, \sigma = 1, \phi(x) = \sin \left(\frac{n \pi}{\ell} x \right).\]

Figure 12: The change of the mean $u$ with time $t$ at $x=1$, $\beta_n = -1$, $h = -0.96$

\[\ell = 1, \sigma = 1, \phi(x) = \sin \left(\frac{n \pi}{\ell} x \right).\]
Figure 13: The change of the variance $u$ at $\varepsilon = 1$, $\beta_n = -1$, $\hat{h} = -0.96$, $(\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).

Figure 14: The change of the variance $u$ with space variable $x$ at $t = 1$, $\beta_n = -1$, $\hat{h} = -0.96$, $(\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).

Figure 15: The change of the variance $u$ with time $t$ at $x = 1$, $\beta_n = -1$, $\hat{h} = -0.96$, $(\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).
Figure 16: The change of the variance $u$ with time $t$ at different $\varepsilon$ values $x=1, \beta_n = -1, h = -.96$ 
$(\ell = 1, \sigma = 1, \phi(x) = \sin \frac{n\pi}{\ell} x)$.

Figure 17: Mean comparison between first $u_1$, second $u_2$ and third order $u_3$ approximations with time $t$ at $x=1, \beta_n = -1, h = -.96$ 
$(\ell = 1, \sigma = 1, \phi(x) = \sin \frac{n\pi}{\ell} x)$.

Figure 18: Mean comparison between first, second and third order approximations $u_1, u_2, u_3$ with space variable $x$ at $t=1, \beta_n = -1, h = -.96$ 
$(\ell = 1, \sigma = 1, \phi(x) = \sin \frac{n\pi}{\ell} x)$.
Figure 19: Variance comparison between first and second approximations $u_1, u_2$ with time $t$ at $x=1$, $\beta_n = -1$, $\lambda = -0.96$, ($\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).

Figure 20: Variance comparison between first and second approximations $u_1, u_2$ with variable $x$ at $t=1$, $\beta_n = -1$, $\lambda = -0.96$, ($\ell = 1$, $\sigma = 1$, $\phi(x) = \sin \frac{n\pi}{\ell} x$).

5 CONCLUSIONS

The homotopy analysis method is employed to give a statistical analytic solution of stochastic quadratic nonlinear diffusion equation. Besides, different from all other analytic methods, the HAM provides us with a simple way to adjust and control the convergence region of the series solution by means of the auxiliary parameter $\hbar$. Thus the auxiliary parameter $\hbar$ plays an important role within the frame of the HAM which can be determined by the so called $\hbar$-curves. The solution obtained by means of the HAM is an infinite power series for appropriate initial approximation, which can be, in turn, expressed in a closed form. The accuracy of the method is verified by comparisons with different methods. As shown in fig.6 and 8 we can see that the valid $\hbar$ region is $-0.9 < \hbar < -1.1$, which is the horizontal line segment and clearly indicate that the HAM gives rapid convergence. The results demonstrate reliability and efficiency of the homotopy analysis method.
REFERENCES


El-Tawil M A and FareedAF (2011), Solution of Stochastic cubic and Quentic Nonlinear Diffusion Using WHEP, Picard and HPM methods. Open Journal of Discrete Mathematics ;1, pp. 6-21


Molabahrami A, Khani F (2009), The homotopy analysis method to solve the Burgers–Huxley equation, Nonlinear Analysis Real World Applications 10, pp. 589–600.


