HOMOTOPY ANALYSIS METHOD FOR SOLVING FRACTIONAL DIFFUSION EQUATION

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ABSTRACT

In this letter, the powerful, easy-to-use and effective mathematical tool like Homotopy Analysis Method is used to solve the diffusion equation with fractional time derivative \( \alpha \) \( (0 < \alpha \leq 1) \). Using the initial condition, the approximate analytical solution of the problem is obtained. Choosing proper values of auxiliary and homotopy parameters, the convergence of the approximate series solution is illustrated for different particular cases. The regions of convergence of the solution for some cases are depicted through graphs.

Keywords: Homotopy analysis method, Fractional diffusion equation, Fractional Brownian motion, Approximate analytical solution.

1 INTRODUCTION

Fractional diffusion equation is obtained from the classical diffusion equation in mathematical physics by replacing the first order time derivative by a fractional derivative of order \( \alpha \) where \( 0 < \alpha < 1 \); of late this being a field of growing interest as evident from literature survey. An important phenomenon of these evolution equations is that it generates the fractional Brownian motion (FBM) which is a generalization of Brownian motion (BM). In 1995, Sebastian has given the correct path integral representation whose measurement shows that the process FBM is Gaussian but in general non-Markovian though BM is Markovian. Due to this, Jumarie (2007) has correctly made the statement that fractional calculus which was in earlier stage considered as a mathematical curiosity, has now become the object of extensive development of fractional partial differential equations for its engineering applications. In the last few decades fractional differential equations are increasingly using in the modeling of various physical and dynamical systems. Recently, Leung and Gou (2011) has successfully used an effective method for solving nonlinear systems having fractional order derivative viz., Forward residue harmonic balance method to obtain analytical approximations to angular frequency and limit cycle for fractional order Van der pol oscillator. The ideas of fractional order derivative are extended even to the nonlinear oscillating systems, which is one of the key areas of research for many years to the engineers and applied mathematicians. Leung and Guo (2010) have successfully solved nonlinear
ordinary differential equation which describes the motion of oscillation with discontinuous and/or fractional power restoring force. Considerable works on fractional diffusion equations have already been done by Angulo et al. (2000), Pezat and Zabczyk (2000), Schneider and Wyss (1989), Yu and Zhang (2006), Mainardi (1996), Mainardi et al. (2001), Anh and Leonenko (2003) etc., using numerical techniques. In view of the restricted applications of prevailing analytical methods and the difficulties posed due to the rounding off errors involved in numerical techniques used by the researchers, the authors have made a sincere effort to explore the solution of the fractional diffusion equation by using a powerful analytical method called Homotopy Analysis Method (HAM).

In 1992, Liao proposed a mathematical tool based on homotopy, a fundamental concept in topology and differential geometry known as Homotopy Analysis Method. It is an analytical approach to get the series solution of linear and nonlinear partial differential equations (PDEs). The difference with the other perturbation methods is that this method is independent of small/ large physical parameters. It also provides a simple way to ensure the convergence of series solution. Moreover the method provides great freedom to choose base function to approximate the linear and nonlinear problems (Liao 1998, 2004). Another advantage of the method is that one can construct a continuous mapping of an initial guess approximation to the exact solution of the given problem through an auxiliary linear operator. To ensure the convergence of the series solution an auxiliary parameter is used. Liao and Tan (2007) have shown that with the help of this method, even complicated nonlinear problems are reduced to the simple linear problems. Recently, Liao (2009) has substantiated the fact that the difference of this method with the other analytical ones is that one can ensure the convergence of the series solution by choosing a proper value of convergence-control parameter. Another important advantage as compared to the other existing perturbation and non-perturbation methods is the freedom to choose proper base function to get the better approximations of the solution to the problems.

The fractional diffusion equation considered here is

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial}{\partial x}(xu(x,t)), \quad 0 < \alpha \leq 1, \ x > 0, \ t > 0$$  \hspace{1cm} (1)$$

with the initial condition

$$u(x,0) = x^n, \quad n \text{ is a positive integer.}$$  \hspace{1cm} (2)$$

Here $\frac{\partial^\alpha}{\partial t^\alpha} (\cdot)$ is the Caputo derivative of order $\alpha$, $u(x,t)$ represents the probability density function at a position $x$ in time $t$. This type of problem is recently solved by (Das 2009) by using another mathematical tool Variational Iteration Method (VIM). Although the method is simple, concise, accurate and effective for linear fractional problems but there are certain drawbacks of the method, namely the restrictions on the order of the nonlinearity term or even the form of the boundary conditions (Tari 2007) and uncontrollability of nonzero end conditions. In this article, a more accurate, flexible and very powerful analytical method HAM is used to solve the equation (1). Using the initial condition (2), the approximate analytical solution of $u(x,t)$ is obtained. As a particular case of the problem for $n = 1$, the
convergence of the series solution with the proper choice of auxiliary parameter and homotopy parameter is shown through Table – 1, which clearly shows validity and potential of the method. The important feature of the article is the investigation of the influences of the auxiliary parameter on the convergence of the solution through $h$-curves analysis.

2 THE BASIC IDEA OF HAM

In this paper, we apply the HAM to the solution of the fractional diffusion problem to be discussed. In order to show the basic idea of HAM, consider the following differential equation

$$N[u(x,t)]=0,$$  

(3)

where $N$ is a non-linear operator, $x$ and $t$ are independent variables, $u(x,t)$ is the unknown function. By means of the HAM, we first construct the so-called zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = \hbar H(x,t)N[\phi(x,t;q)]$$  

(4)

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$, is a nonzero auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator, $u_0(x,t)$ is the initial guess of $u(x,t)$. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, equation (4) becomes $\phi(x,t;0) = u_0(x,t)$ and $\phi(x,t;1) = u(x,t)$ respectively. Thus, as $q$ increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to $q$, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t)q^k$$  

(5)

where $u_k(x,t) = \left. \frac{\partial^k \phi}{\partial q^k} \right|_{q=0}$.  

(6)

The convergence of the series (5) depends upon the auxiliary parameter $\hbar$. If it is convergent at $q=1$, one has

$$\phi(x,t;1) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t)$$

which must be one of the solutions of the original nonlinear equation, as proven by (Liao 2003). Now we define the vector

$$\bar{u}_n(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\},$$  

(7)

So the mth-order deformation equations are

\[ L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h R_m(\tilde{u}_{m-1}(x,t)), \] (8)

with the initial conditions
\[ u_m(x,0) = 0, \] (9)

Where
\[ R_m(\tilde{u}_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}} \bigg|_{q=0}, \]

and \( \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \)

Now, the solution of the mth-order deformation equation (8) for \( m \geq 1 \) becomes
\[ u_m(x,t) = \chi_m u_{m-1}(x,t) + h J^\alpha \{R_m(\tilde{u}_{m-1}(x,t))\} + c, \] (10)

where \( c \) is the integration constants determined by the initial condition (9) and
\[ J^\alpha \{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi. \] In this way, it is easy to obtain \( u_m(x,t) \) for \( m \geq 1 \), at m-th order and finally get the solution as
\[ u(x,t) = \sum_{m=0}^{N-1} u_m(x,t). \] (11)

3 SOLUTION OF THE PROBLEM BY HAM

To solve equation (1) by HAM, we choose the initial approximation
\[ u_0(x,t) = x^\alpha, \] (12)

and the linear operator,
\[ L[\phi(x,t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha}, \] (13)

with the property
\[ L[c] = 0, \] (14)

where \( c \) is integral constant. Furthermore, equation (1) suggests that we define an equation of nonlinear operator as

\[ N[\phi(x, t; q)] = \frac{\partial^\alpha \phi(x, t; q)}{\partial t^\alpha} - \frac{\partial^2 \phi(x, t; q)}{\partial x^2} - \frac{\partial}{\partial x}(x \phi(x, t; q)), \]  

Now, we construct the zero-th order deformation equation

\[ (1 - q)L[\phi(x, t; q) - u_0(x, t)] = q \hbar N[\phi(x, t; q)], \]  

Now proceeding is the previous section, we successively obtain

\[ u_1(x,t) = -\hbar \left[ (n+1)x^n + n(n-1)x^{n-2} \right] \frac{I^\alpha}{\Gamma(\alpha+1)}, \]  

\[ u_2(x,t) = -\hbar (h+1) \left[ (n+1)x^n + n(n-1)x^{n-2} \right] \frac{I^\alpha}{\Gamma(\alpha+1)} + \hbar^2 \left[ (n+1)^2 x^n + 2n^2(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4} \right] \frac{I^{2\alpha}}{\Gamma(2\alpha+1)}, \]

\[ u_3(x,t) = -\hbar (h+1)^2 \left[ (n+1)x^n + n(n-1)x^{n-2} \right] \frac{I^\alpha}{\Gamma(\alpha+1)} + 2\hbar^2 (h+1) \left[ (n+1)^2 x^n + 2n^2(n-1)x^{n-2} + 3n(n-1)(n-2)(n-3)x^{n-4} + n(n-1)(n-2)(n-3)(n-4)(n-5)x^{n-6} \right] \frac{I^{3\alpha}}{\Gamma(3\alpha+1)}, \]

\[ u_4(x,t) = -\hbar (h+1)^3 \left[ (n+1)x^n + n(n-1)x^{n-2} \right] \frac{I^\alpha}{\Gamma(\alpha+1)} + 3\hbar^2 (h+1)^2 \left[ (n+1)^2 x^n + 2n^2(n-1)x^{n-2} + 3n(n-1)(n-2)(n-3)x^{n-4} + n(n-1)(n-2)(n-3)(n-4)(n-5)x^{n-6} \right] \frac{I^{4\alpha}}{\Gamma(4\alpha+1)}, \]
In this manner the rest of the components $u_n$, $n > 4$ of the HAM can be completely obtained and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series

$$u(x,t) = \lim_{N \to \infty} \Phi_N(x,t)$$

(21)

where $\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$, $N \geq 1$

The above series solutions generally converge very rapidly. The rapid convergence means that few terms are required.

4 NUMERICAL RESULTS AND DISCUSSION

In this section, the values of $u(x,t)$ for the initial condition $u(x,0) = x$ for various values of $t$ and $x = 1$ with the proper choices of $h$, $q$ are obtained and the results are depicted in the Table 1. Here sixth order term approximation is taken of the series solution during the numerical computation. As stated by (Liao 2004), it is seen that when $h = -1$, $q = 1$ the result resonates with the result obtained by another mathematical tool Homotopy Perturbation Method (He 2000, 2003). It is seen that for this particular case the result is similar to the result obtained by VIM (Das 2009). It is also seen for the Table 1 that for HAM, the convergence of the values of $u(x,t)$ are found quite similar by controlling the values of auxiliary parameter $h$ and homotopy parameter $q$. This clearly demonstrates the statement of Liao (1992, 1997, 2004) that the method provides great freedom to choose initial approximation, homotopy parameter $q$, the auxiliary linear operator $L$, and the auxiliary parameter $h$ to ensure the convergence of the series solution.

It is also seen from the Table 1 that $u(x,t)$ increases with the increase in $t$ for every $\alpha$ and decreases with the increase of the fractional time derivative $\alpha$. This clearly portages the decay of the probability density function $u(x,t)$ with $\alpha$ ($0 < \alpha < 1$), which is the stress exponential decay characteristic. This is in agreement with the result of Das (2009). The shape is called the stretched Gaussian (Giona and Roman 1992).

Figures 1 and 2 represent the plots of the physical quantity $u_c(x,t)$ which is a function of auxiliary parameter $h$, against $h$ for $\alpha = 0.75$, $\alpha = 1$ and $t = 0.3, 0.7$. Since $u_c(x,t)$ converges to the exact values for different values of $h$, there exists horizontal line segments shown in the figures, which are usually called $h$-curves and this shows the validity of the region of convergence of the series solution (21). The results clearly justify the statement of Liao (2003) that by means of HAM, the convergence region and the rate of series solution can be adjusted and controlled by plotting $h$-curves.

It is also observed from the figures that for smaller time the region of the convergence will be better. It is also observed that in comparison to the standard motion, the convergence region will be less for fractional Brownian motion.

Table 1: Comparison of VIM and HAM results of $u(x, t)$ for different values of $t$ and $\alpha$ at $n = 1$ and $x = 1$.

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Figure 1(a): Plot of $u_{i}(x, t)$ vs. $h$ at $\alpha = 0.75$ and $t = 0.3$
Figure 1(b): Plot of $u_i(x,t)$ vs. $h$ at $\alpha = 0.75$ and $t = 0.7$

Figure 2(a): Plot of $u_i(x,t)$ vs. $h$ at $\alpha = 1$ and $t = 0.3$

Figure 2(b): Plot of $u_i(x,t)$ vs. $h$ at $\alpha = 1$ and $t = 0.7$

5 CONCLUSION

The main interest of this study is successful implementation of the powerful mathematical tool HAM to investigate the solution of the evolution equation and to show its efficacy in contrast to the other reliable mathematical tools like HPM and VIM. This method provides us a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. Thus it may be concluded that HAM is a simple and a powerful analytical approach for handling fractional related PDEs.

As in agreement with the previous work (Das 2009), the decrease of the values of $u(x,t)$ with the increase of fractional time derivatives is observed. However faster computation procedure of the present method and the convergence criterion of the series solution with the proper choices of auxiliary and homotopy parameters render the article a different dimension and the authors strongly believe that the present article will be highly acceptable by the researchers working in the field of fractional calculus.

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REFERENCES


