ON THE CLASS OF THREE-DIMENSIONAL UNSTEADY INCOMPRESSIBLE BOUNDARY LAYER EQUATIONS OF NON-NEWTONIAN POWER LAW FLUIDS

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ABSTRACT

Nonlinear partial differential equations governing the motion of three-dimensional unsteady incompressible laminar boundary layer flow of non-Newtonian power-law fluid past a flat plate are treated. The governing equations containing three independent variables are reduced to the system of highly nonlinear ordinary differential equations, by two parameter group of transformations in a single step. All possible conditions under which the similarity solutions for the present flow situation exists are automatically derived from the similarity requirements and thus the similarity solution is found in most general form.

Keywords: Non-Newtonian power-law fluid, Similarity solution, two-parameter group of Transformations

1 INTRODUCTION

It was Prandtl (Prandtl 1928) who has first introduced the concept of boundary-layer in fluid mechanics, and as a result a great deal of work, both analytical and experimental, has been directed towards its application. The first analytical application of Prandtl’s theory was given by Blausis (Blausis 1908), in his investigation of the flow of an infinite uniform stream over a thin flat plate at zero incidences. In the last decade Oleinik and Samokhin (Oleinik and Samokhin 1999) have studied a lot of exact results concerning the boundary layer equations of pseudo-plastic fluids. Following Oleinik et al (Oleinik et al. 1999), Polyanin and Zaifsev (Polyanin and Zaifsev 2001, 2004) have also contributed much to the development of the application of boundary layer equations of Newtonian or non-Newtonian fluids.

Power-law fluids are by far the most widely used model to exhibit non-Newtonian behavior in fluids, and to predict shear thinning and shear thickening behavior. However, it has an inadequacy in expressing normal stress behavior as observed in die swelling and rod climbing behavior in some non-Newtonian fluids. On the other hand, normal stress effects can be expressed in a second grade fluid model, a special type of Rivlin-Ericksen fluids, but this model is incapable of representing shear thinning/thickening behavior. Due to the academic

curiosity and industrial demand, in the second half of the last century boundary layer of non-Newtonian fluids has received considerable attention. This theory has been applied successfully to various non-Newtonian fluid models. For power law fluids some of the previous work is due to Acrivos et al (Acrivos et al. 1960), Schowalter (Schowalter 1960), Lee and Ames (Lee and Ames 1966), Hansen and Na (Hansen and Na 1967), Na and Hansen (Na and Hansen 1968), Timol and Kalthia (Timol and Kalthia 1986), Pakdemirli (Pakdemirli 1993). The shear thinning boundary layer flow of power law fluid was investigated by Mansutti and Rajagopal (Mansutti and Rajagopal 1991). The boundary layers they studied are conceptually different from the usual boundary layers. For the second grade fluid the early work includes Srivastava (Srivastava 1958), Rejeswari and Rathna (Rejeswari and Rathna 1962), while more recent work includes Rajagopal et al (Rajagopal et al. 1980,1986), Garg and Rajagopal (Garg and Rajagopal 1991), Pakdemirli and Suhubi (Pakdemirli and Suhubi 1992). Boundary layer equations for third grade fluids are studied by Pakdemirli (Pakdemirli 1992) using a convenient coordinate system.

Lee and Ames (Lee and Ames 1966) were probably first to derive similarity analysis of boundary layer equations of all those non-Newtonian fluids which are characterized by the explicit functional relationship between shearing stress and symmetric part of the velocity gradient. The work of Lee and Ames (Lee and Ames 1966) was immediately extended by the Hansen and Na (Hansen and Na 1967) by reconsidering the same relationship as some arbitrary implicit function. Timol and Kalthia (Timol and Kalthia 1986) have extended the work of Hansen and Na (Hansen and Na 1967) for three dimensional flows of various non-Newtonian fluids. Pakdemirli (Pakdemirli 1994) have extended the analysis of previous work Pakdemirli (Pakdemirli 1993) by finding special form of shear stress with more general similarity representation.

The group theoretic method is of wide applicability and is a well accepted method to find the similarity solutions in many physical situations. It was first reported by Birkhoff (Birkhoff 1955) and later a number of authors like Morgan (Morgan 1952), Hansen (Hansen 1965), Na (Na 1982), and Seshadri and Na (Seshadri and Na 1985), have contributed much to the development of the theory. The method has been applied intensively by Hansen and Na (Hansen and Na 1967), Pakdemirli (Pakdemirli 1994), Fan et al (Fan et al. 1998), Abd-el-Malek and Badran (Abd-el-Malek and Badran 1990), Abd-el-Malek et al (Abd-el-Malek et al 2002). In this paper similarity analysis is made of three-dimensional unsteady incompressible laminar boundary-layer flow of non-Newtonian power-law fluids past a flat plate. By applying two-parameter group of transformations the set of governing equations and the boundary conditions are reduced to highly nonlinear ordinary differential equations with appropriate boundary conditions in a single step. The outcome of this study is that, all possible conditions under which the similarity solutions for the present flow situation exists are automatically derived from the similarity requirements and thus the similarity solution is found in most general form.

2 PROBLEM FORMULATION

In order to describe any fluid, it must be possible to write its stress tensor \( \tau \), as a function of known variables. For Newtonian fluids, this is easily done:

\[
\tau = \mu \Delta
\]

(1)

where \( \Delta \) is the “rate of deformation tensor” with Cartesian components.

\[
\Delta_{ij} = \left( \frac{\partial \nu_i}{\partial x_j} \right) + \left( \frac{\partial \nu_j}{\partial x_i} \right)
\]

and \( \mu \) is the coefficient of viscosity.

The coefficient of viscosity depends on the local temperature and pressure but not on \( \tau \) or \( \Delta \).

The formulation of \( \tau \) for non-Newtonian fluids is, in general, a complex problem and awaits a well-developed theoretical basis. The description of non-Newtonian fluids is also hampered by the lack of experimental techniques to evaluate various analytical models. Two examples of general fluid models which are based on theoretical considerations but lack sufficient experimental information to relate them to real fluids are: (1) a three-constant model suggested by Oldroyd (Oldroyd 1956) for non-Newtonian fluids which exhibit elasticity, and (2) a model for non-Newtonian fluids that do not exhibit elasticity which features two scalar functions of \( \Delta \) the effective viscosity and the “cross viscosity” Rivlin (Rivlin 1948) and Reiner (Reiner 1945). Lacking a usable model based on theoretical considerations, it is necessary to formulate phenomenological models, which can be used in the equations of motion. These models must, however, obey the laws of tensor transformation. Keeping this in mind, it is possible to relate \( \tau \) and \( \Delta \) by means of an effective viscosity Bird (Bird 1960), for fluids which are purely viscous (no elasticity or anisotropic normal stresses), by

\[
\tau = \mu_e \Delta
\]

where \( \mu_e \), a scalar is a function of \( \Delta \) as well as a function of temperature and pressure. In order for \( \mu_e \) to be a scalar function of the tensor \( \Delta \) it must depend only on the invariants of \( \Delta \), which are,

\[
I_1 = \sum_i \Delta_{ii}
\]

\[
I_2 = \sum_i \sum_j \Delta_{ij} \Delta_{ji}
\]

\[
I_3 = \sum_i \sum_j \sum_k \varepsilon_{ijk} \Delta_{1i} \Delta_{2j} \Delta_{3k}
\]

\( I_1 \) can be shown to be equal to zero for an incompressible fluid and \( I_3 \) is identically zero for a non-Newtonian fluid characterized by equation (2), the stress tensor is given by

\[
\tau = \mu_e \left( I_2 \right) \Delta
\]

At this point several choices of empirical relations for \( I_2 \) are available. Due to this convenient form and the fact that it represents a large number of fluids very well, the Ostwald-de Waele model is chosen:

\[ \mu_e (I_2) = a \sqrt[2n-1]{\frac{1}{2} \Delta : \Delta} \quad (n \geq 0) \]

where \( \Delta : \Delta \) denotes the scalar product of two tensors. For three dimensional flow, following Na and Hansen (Na and Hansen 1967) equation (4) can be combined with equation (3) to give an expression for the shear stress to be considered in the boundary layer equations:

\[ \tau_{yx} = -a \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} \frac{\partial u}{\partial y} \]

\[ \tau_{yz} = -a \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} \frac{\partial w}{\partial y} \]

where \( \tau_{yx} \) and \( \tau_{yz} \) are the shear stress in the x and z-directions respectively due to a velocity gradient in the y-direction and \( a \) and \( n \) are physical constants different for different fluids which can be determined experimentally. Further the absolute sign has been dropped since both terms within the sign are positive. The role flow consistency index \( n \) is crucial, as for \( n < 1 \), the flow can be described as shear thinning; and for \( n > 1 \), the flow is shear thickening. For \( n = 1 \) the expression describes a Newtonian fluid with \( a = \mu \).

3 ANALYSIS

As mentioned above, the procedure will be to normalize the fluid velocity and the temporal and spatial coordinates and, using the classical boundary layer equations, to write the partial differential equation for the stream function. The problem then will be that of determining the form of a single variable which will allow the transformation of the partial differential equation for the stream function in to an ordinary differential equation in terms of a new stream function and the single variable. Once this equation and the appropriate boundary conditions are written, and the significance of the individual terms noted, it will be possible to systematically consider the various conditions which must be satisfied to obtain similar solutions. The system of equations considered are those credited to Prandtl (Prandtl 1928) and must therefore being accompanied by his postulation of a thin boundary layer. For an incompressible, laminar, three-dimensional, thin boundary layer without regard to the form of the momentum flux term, the continuity and momentum equations are:

**Continuity Equation**

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]
Momentum Equation

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \frac{n-1}{2} \frac{\partial u}{\partial y} + U \frac{dU}{dx} + \frac{dU}{dt} \]  

(8)

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \frac{n-1}{2} \frac{\partial w}{\partial y} + U \frac{dW}{dx} + \frac{dW}{dt} \]  

(9)

with the boundary conditions,

\[ y = 0: \quad u = v = w = 0 \]

\[ y = \infty: \quad u = U(x, t), \quad w = W(x, t) \]  

(10)

where the dimensionless quantities used are,

\[ u = \frac{u'}{U_0}, \quad v = \frac{v'}{U_0} \text{Ren}^{n+1}, \quad w = \frac{w'}{U_0}, \quad U = \frac{U'}{U_0} \]

\[ W = \frac{W'}{U_0}, \quad x = \frac{x'}{L}, \quad y = \frac{y'}{L} \text{Ren}^{n+1}, \quad t = \frac{t'}{b}, \quad b = \frac{L}{U_0} \]

where \( \text{Ren} = \frac{\rho U_0^{2-n} L^n}{m}, \quad m = 2^{-(n+1)/2} \) and \( K \) is the consistency of the fluid.

The flow problem is quasi-two-dimensional in nature since the velocity components are independent of the z-coordinates. It is hoped that by assuming independence of flow quantities in one direction, more quantitative information may be obtained on the characteristics of three-dimensional boundary-layer flows. The equation of continuity can be satisfied identically by introducing a function \( \psi \), which gives

\[ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \]

Equations (7), (8), (9) and (10) becomes
\[
\frac{\partial^2 \psi}{\partial \psi \partial \gamma} + \frac{\partial \psi}{\partial \gamma} \frac{\partial^2 \psi}{\partial x \partial \psi} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial \psi^2} = \frac{\partial}{\partial \gamma} \left[ \left( \frac{\partial \psi}{\partial \gamma} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2 \right]^{n-1} \frac{2}{2} \frac{\partial^2 \psi}{\partial \gamma^2} + \frac{\partial U}{\partial x} + \frac{dU}{dt} \tag{11}
\]

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial \gamma} \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial \gamma} \frac{\partial \psi}{\partial \gamma} = \frac{\partial}{\partial \gamma} \left[ \left( \frac{\partial \psi}{\partial \gamma} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2 \right]^{n-1} \frac{2}{2} \frac{\partial \psi}{\partial \gamma} + \frac{\partial W}{\partial x} + \frac{dW}{dt} \tag{12}
\]

with the boundary conditions,

\[
y = 0: \quad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = w = 0
\]

\[
y = \infty: \quad \frac{\partial \psi}{\partial y} = U(x, t), \quad w = W(x, t) \tag{13}
\]

A group-theoretic analysis is employed in the next section to find the specific form of free stream velocities \( U(x, t) \) and \( W(x, t) \) for which similarity solutions will exist.

### 4 GROUP-THEORETIC ANALYSIS

Exact solution of boundary layer equations have mainly been found by using the similarity technique. In this technique the partial differential equations of the boundary layer are reduced to ordinary differential equations which are then usually solved numerically. Solutions obtained by similarity techniques are known as similar solutions. In this technique flows are restricted to the basic assumption that the velocity profile varies at most by a scale factor along the coordinate lines. In view of this, as might well be expected the type of coordinate system used governs to a great extent the types of flows for which exact similar solutions can be obtained. There are number of research papers available on various similarity techniques. Among all these techniques Group theoretic method discussed by Hansen (Hansen 1965), Seshadri and Na (Seshadri and Na 1985) are most elegant and successful techniques due to its simplicity and applicability. Group theoretic method is based on the basic concepts of linear group of transformation. By this method one can reduced number of independent variables from the system of partial differential equation at a time.

Consider first, the two-parameter group transformation defined by,

\[
G_{21} : \quad t = A^1 \bar{t}, \quad x = B^1 \bar{x}, \quad y = A^3 B^3 \bar{y}, \quad \psi = A^4 B^4 \bar{\psi}
\]

\[
w = A^5 B^5 \bar{w}, \quad U = A^6 B^6 \bar{U}, \quad W = A^7 B^7 \bar{W}
\]

where \( \alpha_i \) and \( \beta_i \) (\( i = 1,2,\ldots,7 \)) and A, B are constants.
We now seek relations among the $\alpha_i$’s and $\beta_i$’s such that the basic equations (11) and (12) will be invariant under this group of transformation. This can be achieved by substituting the group transformation $G_{21}$ into equations (11) and (12). Thus, we obtain

\[
\alpha_4^2 - \alpha_3^2 - B^{2} \beta_4 - \beta_3 \frac{\partial^2 \psi}{\partial \alpha \partial y} + A^{2} \alpha_4^2 - 2 \alpha_3^2 B^{2} \beta_4 - 2 \beta_3 - \beta_1 \left[ \frac{\partial^2 \psi}{\partial y \partial \alpha} - \frac{\partial^2 \psi}{\partial \alpha \partial x} \right] = \frac{\partial}{\partial y} \left[ \frac{n \alpha_4^4 - (n+1) \alpha_3^4}{B^{4}} + \frac{n \beta_5^4 - (n+1) \beta_3^4}{B^{4}} \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2 \right] + A^{2} \alpha_5^4 + A^{4} \alpha_5^4 - \alpha_3^4 B^{4} + 2 \beta_5^4 - 2 \beta_3^4 - \beta_1 \left[ \frac{\partial^2 \psi}{\partial y \partial \beta} - \frac{\partial^2 \psi}{\partial x \partial \beta} \right]
\]

And

\[
\alpha_5^2 - \alpha_3^2 - B^{2} \beta_5 + A^{2} \alpha_5^2 - 3 \beta_4 - \beta_3 - \beta_1 \left[ \frac{\partial^2 \psi}{\partial y \partial \alpha} - \frac{\partial^2 \psi}{\partial \alpha \partial x} \right] = \frac{\partial}{\partial y} \left[ \frac{(n-1) \alpha_4^4 - 2 n \alpha_3^4}{B^{4}} + \frac{(n-1) \beta_5^4 - 2 n \beta_3^4}{B^{4}} \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2 \right] + A^{2} \alpha_5^4 + A^{4} \alpha_5^4 - \alpha_3^4 B^{4} + 2 \beta_5^4 - 2 \beta_3^4 - \beta_1 \left[ \frac{\partial^2 \psi}{\partial y \partial \beta} - \frac{\partial^2 \psi}{\partial x \partial \beta} \right]
\]

From equations (14) and (15), it is seen that if the basic equations are to be invariant under this group of transformation, we obtain the relations among the $\alpha_i$’s and $\beta_i$’s as,

\[
\frac{\alpha_3}{\alpha_1} = \frac{2 - n}{n + 1}
\]

\[
\frac{\alpha_4}{\alpha_1} = \frac{1 - 2 n}{n + 1}
\]

\[
\frac{\alpha_5}{\alpha_1} = \frac{\alpha_6}{\alpha_1} = \frac{\alpha_7}{\alpha_1} = -1
\]
The next step in this method is to find the so-called “absolute invariants” under this group of transformation. Absolute invariants are functions having the same form before and after the transformation. The absolute invariants are,

\[ \eta = \frac{y}{\left(2 - \frac{2n}{n + 1}\right)} \]

\[ F_1(\eta) = \frac{\psi}{\left(1 - \frac{2n}{n + 1}\right) - \frac{2n}{x(n + 1)}} \]

\[ G_1(\eta) = \frac{w}{t^{-1} x} \]

\[ C_1 = \frac{U}{t^{-1} x} \]

\[ C_2 = \frac{W}{t^{-1} x} \]

Here equations (25-26) are due to the invariance of boundary conditions (13). Substituting these quantities from equations (22-26) in the basic equations we obtain a set of ordinary differential equations as,

\[ \frac{d}{d\eta} \left[ \left( \frac{d^2 F_1}{d\eta^2} \right)^2 + \left( \frac{dG_1}{d\eta} \right)^2 \right]^{n-1} \frac{d^2 F_1}{d\eta^2} \left( \frac{2}{n+1} \right) \eta \frac{d^2 F_1}{d\eta^2} - \left( \frac{dF_1}{d\eta} \right)^2 \right) \]

\[ + \frac{2n}{(n + 1)} F_1 \frac{d^2 F_1}{d\eta^2} + C_1 (C_1 - 1) = 0 \]
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\[ \frac{d}{d\eta} \left( \left[ \left( \frac{d^2 F_1}{d\eta^2} \right) \right]^2 + \left( \frac{dG_1}{d\eta} \right)^2 \right)^{\frac{n-1}{2}} \frac{dG_1}{d\eta} + G_1(\eta) + \left( \frac{2-n}{n+1} \right) \eta \frac{dG_1}{d\eta} + \frac{2n}{(n+1)} F_1 \frac{dG_1}{d\eta} \]

with the transformed boundary conditions,

\[ \eta = 0 : \quad F_1 = F_1' = G_1 = 0 \]
\[ \eta = \infty : \quad F_1' = C_1, \quad G_1 = C_2 \]

The solutions \( F_1(\eta) \) and \( G_1(\eta) \) of (27-28) together with their boundary conditions (29), by uniqueness satisfies the identity,

\[ C_1 G_1 = C_2 F_1' \]

where \( F_1(\eta) \) is either null or is the self-similar part of the stream function for the boundary layer equations of a power law fluid, whose consistency is

\[ m' = m \left( 1 + \left( \frac{C_2}{C_1} \right)^2 \right)^{(n-1)/2} \]

with no transverse flow. Further the similarity equations (27) and (28) together with their boundary conditions (29), for two dimensional cases, for some particular values of parameters are solved numerically by Patel and Timol (Patel and Timol 2009).

5 CONCLUSIONS

The important conclusion drawn from this analysis is that for the case of three-dimensional unsteady incompressible laminar boundary-layer equations of non-Newtonian power-law fluids, the similarity solutions for the present flow situation exists are automatically derived from the similarity requirements and thus the similarity solution is found in most general form by using two-parameter group of transformation in a single step.

NOTATIONS

- \( \tau \) : Shear stress
- \( \Delta \) : The rate of deformation tensor
- \( \mu \) : Viscosity
\[ \mu_e : \text{Effective viscosity} \]

\[ \rho : \text{Density of fluid} \]

\[ I_1, I_2, I_3 : \text{Invariants of } \mathbf{\Delta} \]

\[ a, n : \text{Parameters of power-law fluid model} \]

\[ \tau_{yx} : \text{The non-vanishing component of the stress tensor.} \]

\[ u, v, w : \text{Velocity components along the x, y and z-axes} \]

\[ t : \text{Time} \]

\[ U(x, t), W(x, t) : \text{Main stream velocity in x and z-direction} \]

\[ L : \text{A characteristic length} \]

\[ \text{Ren} : \text{Reynolds number} \]

\[ \psi : \text{A mathematical function} \]

\[ F_1, G_1 : \text{Dependent variables in the transformed ordinary differential equations} \]

\[ \eta : \text{Independent variable in the transformed ordinary differential equations} \]

\[ \alpha, \beta_i (i = 1, 2, ..., 7), C_1, C_2, A \text{ and } B : \text{Arbitrary constants} \]

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