APPROXIMATION OF BURGERS’ EQUATION USING B-SPLINE
FINITE ELEMENT METHOD

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ABSTRACT
In this paper we present a numerical scheme for solving Burgers equation using B-spline finite element method. Main advantages of the present technique are: it is simple, efficient and moreover easy to understand. Some numerical examples are considered to test the functioning of proposed scheme. Results obtained from the given method of solution are validated by means of comparisons made with the solutions available in the literature. We observe that the numerical solutions coincided with the exact solution and the agreement between numerical and exact solutions appeared very satisfactory.

Keywords: Burger equation, B-spline functions, Finite Element Method.

1 INTRODUCTION
Mathematical modelling of many frontier physical systems leads to nonlinear differential equations, e.g., Burgers equation, Korteweg-de Vries(KdV) equation, Klein-Gordon equation, equation, etc. For better understanding and application of these phenomena demands their solutions. Due to the fact that analytical solutions do not exist for some engineering problems, we need to develop some fast and accurate numerical schemes to analyze these mathematical models. Many researchers are involved in finding the solution of such model problems (see e.g., (C.M.Khalique 2009; C.M.Khalique 2008; Dogan 2004; T. Öziş 2003; Pöschl 1997; P.A.Lagerstrom 1949)).

Finite element method is a well established numerical technique. It was originally employed for structural analysis only but over four decades, it is now widely being applied for different problems in solid mechanics, heat transfer and fluid mechanics (Shabani and Mazahery 2011; Aksan 2005; O. Anwar Beg and Beg 2011; S. K. Ghosh and Ghani 2010). We observe from the literature that a close link between geometric modelling and numerical simulation in Int. J. of Appl. Math. and Mech. 7(17): 61-86, 2011.
engineering applications has suggested the use of B-splines as finite element basis functions. Those who undergo the task of implementing B-splines in finite elements typically note the increased smoothness, accuracy, and computational savings (Pöschl 1997; S.Wendel 1993) compared to conventional finite element. De Boor’s contributions about B-splines play an important role in numerical approximation (de Boor 1978). B-Spline curves were created as an improvement over Bezier curves in the 1970’s. They were first introduced by Schoenberg in (I.J.Schoenberg 1946; I.J.Schoenberg 1969) and many of their algebraic properties can be found in (H.B.Curry and I.J.Schoenberg 1966). Also, they require less time and effort in computing (Fuhua Cheng 2001). Some researchers have used these functions (Mittal and Arora 2010; Dogan 2004; S. Kutluay 2004; Kapoor and Dhawan 2010) to find solution of variety of problems. In the present work, Petrov-Galerkin B-spline Finite Element Method has been used to solve the Burger equation numerically.

Burgers equation first appeared in 1915 in a paper by Bateman (H.Bateman 1915), where he used this equation as a model for the motion of a viscous fluid when the viscosity approaches zero. Later in a remarkable series of papers from 1939 – 1965, Burger investigated various aspects of turbulence and developed a mathematical model illustrating the theory as well as statistical and spectral aspects of the equation and related systems (J.M.Burgers 1990; J.M.Burgers 1948; J.M.Burgers 1972). Due to extensive work of Burger, it is known as Burgers equation. It plays an important role in studying various problems of science and engineering such as shock flows, growth of molecular interfaces, traffic flows, nonlinear wave propagation/shock waves, gas dynamics, longitudinal elastic waves in an isotropic solid, turbulence (M.Vergassola 1994; S.A.Molchanov 1995) and so forth. The nonlinear term \( \frac{\partial U}{\partial x} \) makes it more interesting to study as it is one of the few non-linear partial differential equations that have been solved analytically. Burgers equation has attracted many researchers from the past many years (N. Nguyen 1982; Mittal and P.Singhal 1993; T. Özis 2003; Turgut Özis 2005). In this paper, we use both linear as well as quadratic B-spline functions for the numerical simulation of Burgers equation

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = \nu \frac{\partial^2 U}{\partial x^2},
\]

with the given initial and boundary conditions

\[
U(x,0) = f(x), \quad t > 0,
\]
\[
U(a,t) = U(b,t) = 0, \quad a \leq x \leq b, \quad t > 0.
\]

Some test examples are considered to find the solution using proposed method of solution and results are found to be satisfactory. An easy way of pursuing the whole numerical simulation is explained in detail.

2 SOLUTION PROCEDURE

The interval \([a, b]\) is divided into \(N\) finite elements of equal length \(\Delta x = x_{m+1} - x_m\) with \(x_m\) as knots such that \(a = x_0 < x_1 < \ldots < x_N = b, m = 0, 1, 2, \ldots, N - 1\). On this partition we will use linear and quadratic B-splines with knots at points \(x_i, i = 0, 1, \ldots, N\). The least square formulation for equation (1) gives

\[
\delta \int_a^b \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} - \nu \frac{\partial^2 U}{\partial x^2} \right)^2 \, dx \, dt = 0.
\]
Introducing the normalized variables \( \eta, \tau \) such that
\[
\eta = \frac{x - x_m}{\Delta x}, \quad \tau = \frac{t - t_n}{\Delta t}, \quad 0 \leq \eta \leq 1, \quad 0 \leq \tau \leq 1,
\]
(5)

Using the above transformation, equation (4) over each typical element \([x_m, x_{m+1}]\) can be written as
\[
\delta \int_0^1 \int_0^1 \left( \frac{\partial U}{\partial \tau} + U^* \frac{\Delta t}{\Delta x} \frac{\partial U}{\partial \eta} - \nu \frac{\Delta t}{(\Delta x)^2} \frac{\partial^2 U}{\partial \eta^2} \right)^2 d\eta d\tau = 0.
\]
(6)

where \( U^* \) is taken as a constant over each element. The integral equation takes its minimum value with the variation in \( U \) over each element \([x_m, x_{m+1}]\). Applying variational principle and taking \( \alpha = U^* \frac{\Delta t}{\Delta x} \), \( \beta = \nu \frac{\Delta t}{(\Delta x)^2} \), equation (6) becomes
\[
\int_0^1 \int_0^1 \left( \frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2} \right) \delta \left( \frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2} \right) d\eta d\tau = 0.
\]
(7)

The least square method turns to Petrov-Galerkin method with the weight function \( \delta(\frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2}) \). Next, we use linear and Quadratic B-spline functions to proceed with the solution procedure. We consider the Burgers equation (1) with the given boundary conditions. A uniform partition of \([a, b]\) as \( a = x_0 < x_1 < \ldots x_N = b \) is introduced and let \( h = (b - a)/N \). On this partition we have points \( x_i, i = 0, 1, \ldots N \).

2.1 Linear B-spline approach (LBA)

The linear B-spline functions which form basis over the solution domain, are defined as (P.M.Prenter 1975)
\[
B_m(x) = \frac{1}{h} \begin{cases} 
  x - x_{m-1}, & \text{if } x \in [x_{m-1}, x_m], \\
  x_{m+1} - x, & \text{if } x \in [x_m, x_{m+1}], \\
  0, & \text{otherwise.}
\end{cases}
\]
(8)

Identifying each finite element in the interval \([x_m, x_{m+1}]\) with nodes at \( x_m \) and \( x_{m+1} \), we have linear B-spline functions over the element \([x_m, x_{m+1}]\) as
\[
\psi_m = 1 - \eta, \quad \psi_{m+1} = \eta, \quad 0 \leq \eta \leq 1.
\]
(9)

On each interval, local approximations are taken of the form
\[
U_N(\eta, \tau) = \sum_{j=m}^{m+1} \psi_j(\eta) (\sigma_j^n + \tau \Delta \sigma_j^n),
\]
(10)

where \( \sigma_j^n \) are nodal parameters at the beginning of the time steps and \( \Delta \sigma_j^n \) are the increments. \( \psi_j \) are B-spline shape functions defined by the local coordinate system having values on each element \([x_m, x_{m+1}]\) as (9). When we use linear B-spline shape functions in the local approximation (10), gives us \( \delta U = \sum_{j=m}^{m+1} \psi_j(\eta) \tau \Delta \sigma_j^n \), and weight function takes the form \( (\psi_i + \alpha \tau \psi_i') \). Substituting in (7), Our least square method turns into Petrov-Galerkin method. Thus we have
\[
\int_0^1 \int_0^1 \left( \frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2} \right) (\psi_j + \alpha \tau \psi_j') d\eta d\tau = 0.
\]
(11)

Splines functions act like shape functions for each element when we set up our equations in terms of element parameter $\sigma_j$. Using (10) in (11), integrating the normalized term and collecting the terms containing $\Delta \sigma_j$ and $\sigma_j$, we get

$$
\sum_{j=m}^{m+1} \left\{ \int_0^1 \left[ \psi_i \psi_j + \frac{\alpha}{2} (\psi_i \psi_j' + \psi_j \psi_i') + \left( \frac{\alpha^2}{3} + \frac{\beta}{2} \right) \psi_i \psi_j \right] d\eta \right. \\
- \frac{\beta}{2} \left( \psi_i \psi_j \right) \left|_0^1 \right. \left\} \Delta \sigma_j^n + \sum_{j=m}^{m+1} \left\{ \int_0^1 \left[ \alpha \psi_i \psi_j' + \beta \psi_i \psi_j' + \frac{\alpha^2}{2} \psi_i \psi_j' \right] d\eta \\
- \frac{\beta}{2} \left( \psi_i \psi_j' \right) \left|_0^1 \right. \left\} \sigma_j^n = 0,
\right.
$$

(12)

Expanding for $i, j = m, m + 1$ gives us

$$
\left\{ \int_0^1 \left[ \psi_m \psi_m + \frac{\alpha}{2} (\psi_m \psi_m' + \psi_m' \psi_m) + \left( \frac{\alpha^2}{3} + \frac{\beta}{2} \right) \psi_m \psi_m' \right] d\eta \right. \\
- \frac{\beta}{2} \left( \psi_m \psi_m' \right) \left|_0^1 \right. \left\} \Delta \sigma_m^n + \left\{ \int_0^1 \left[ \psi_{m+1} \psi_m + \frac{\alpha}{2} (\psi_{m+1} \psi_m' + \psi_{m+1}' \psi_m) \right] d\eta \\
+ \left( \frac{\alpha^2}{3} + \frac{\beta}{2} \right) \psi_{m+1} \psi_m' \right|_0^1 \left\} \Delta \sigma_{m+1}^n \\
+ \left\{ \int_0^1 \left[ \alpha \psi_m \psi_m' + \beta \psi_m \psi_m' + \frac{\alpha^2}{2} \psi_m \psi_m' \right] d\eta \\
- \frac{\beta}{2} \left( \psi_m \psi_m' \right) \left|_0^1 \right. \left\} \sigma_m^n \\
+ \left\{ \int_0^1 \left[ \alpha \psi_{m+1} \psi_m' + \beta \psi_{m+1} \psi_m' + \frac{\alpha^2}{2} \psi_{m+1} \psi_m' \right] d\eta \\
- \frac{\beta}{2} \left( \psi_{m+1} \psi_m' \right) \left|_0^1 \right. \left\} \sigma_{m+1}^n = 0,
\right.
$$

and

$$
\left\{ \int_0^1 \left[ \psi_m \psi_{m+1} + \frac{\alpha}{2} (\psi_m \psi_{m+1}' + \psi_{m+1}' \psi_m) + \left( \frac{\alpha^2}{3} + \frac{\beta}{2} \right) \psi_m \psi_{m+1}' \right] d\eta \right. \\
- \frac{\beta}{2} \left( \psi_m \psi_{m+1}' \right) \left|_0^1 \right. \left\} \Delta \sigma_m^n + \left\{ \int_0^1 \left[ \psi_{m+1} \psi_{m+1} + \frac{\alpha}{2} (\psi_{m+1} \psi_{m+1}' + \psi_{m+1}' \psi_{m+1}) \right] d\eta \\
+ \left( \frac{\alpha^2}{3} + \frac{\beta}{2} \right) \psi_{m+1} \psi_{m+1}' \right|_0^1 \left\} \Delta \sigma_{m+1}^n \\
+ \left\{ \int_0^1 \left[ \alpha \psi_m \psi_{m+1}' + \beta \psi_m \psi_{m+1}' + \frac{\alpha^2}{2} \psi_m \psi_{m+1}' \right] d\eta \\
- \frac{\beta}{2} \left( \psi_m \psi_{m+1}' \right) \left|_0^1 \right. \left\} \sigma_m^n \\
+ \left\{ \int_0^1 \left[ \alpha \psi_{m+1} \psi_{m+1}' + \beta \psi_{m+1} \psi_{m+1}' + \frac{\alpha^2}{2} \psi_{m+1} \psi_{m+1}' \right] d\eta \\
- \frac{\beta}{2} \left( \psi_{m+1} \psi_{m+1}' \right) \left|_0^1 \right. \left\} \sigma_{m+1}^n = 0.
\right.
$$

Defining element matrices

$$
X_{i,j}^n = \int_0^1 (\psi_i \psi_j) d\eta, \quad Y_{i,j}^n = \int_0^1 (\psi_i' \psi_j') d\eta, \quad Z_{i,j}^n = \int_0^1 (\psi_i \psi_j') d\eta
$$

(13)

and using these element matrices $X^e, Y^e, Z^e$ above formulation can be expressed as

$$
\begin{align*}
\left\{ X^e + \left( \frac{\alpha^2}{3} + \frac{\beta}{2} \right) Y^e + \left( \frac{\alpha}{2} (Z^e + (Z^e)^T) \right) - \frac{\beta}{2} \left( \psi_i \psi_j' \right) \right\}_0^1 \Delta \sigma_j^n &= 0, \\
+ \left\{ \left( \frac{\alpha^2}{2} + \beta \right) Y^e + \alpha Z^e - \beta \left( \psi_i \psi_j' \right) \right\}_0^1 \sigma_j^n &= 0.
\end{align*}
$$

Assembling together contributions from all the elements and using $\Delta \sigma_j^n = \sigma_{j+1}^n - \sigma_j^n$ and $\sigma = \sigma_1^n$, we get

$$
\begin{align*}
\left\{ X + \left( \frac{\alpha^2}{3} + \frac{\beta}{2} \right) Y + \left( \frac{\alpha}{2} (Z + Z^T) \right) - \frac{\beta}{2} \left( \psi_i \psi_j' \right) \right\}_0^1 \sigma_{j+1}^n &= 0, \\
+ \left\{ \left( \frac{\alpha^2}{2} + \beta \right) Y + \alpha Z - \beta \left( \psi_i \psi_j' \right) \right\}_0^1 \sigma_j^n &= 0.
\end{align*}
$$

A typical member of the matrix system (15) can be expressed in terms of nodal parameter $\sigma$ as:

$$
\begin{align*}
\kappa_1 \sigma_{m-1}^{n+1} + \kappa_2 \sigma_m^{n+1} + \kappa_3 \sigma_{m+1}^{n+1} &= \kappa_4 \sigma_m^n + \kappa_5 \sigma_{m+1}^n + \kappa_6 \sigma_{m+1}^n, \\
\end{align*}
$$

where

$$
\begin{align*}
\kappa_1 &= \frac{1}{6} - \frac{\alpha^2}{2}, \quad \kappa_2 = \frac{2}{3} + \alpha^2, \quad \kappa_3 = \frac{1}{6} - \frac{\alpha^2}{2}, \\
\kappa_4 &= \frac{1}{6} + \frac{\alpha}{2}, \quad \kappa_5 = \frac{4}{6}, \quad \kappa_6 = \frac{1}{6} - \frac{\alpha}{2}.
\end{align*}
$$

We write $L$ for left hand matrices and $R$ for right hand matrices and express (15) in a compact form as

$$
[L] \sigma_{m+1}^{n+1} = [R] \sigma_m^n.
$$

Approximate solution to the given Burgers equation (1) is obtained from this recurrence relationship. Process is repeated once the initial approximation is obtained using the given initial condition. Taking initial value from the given initial condition, we can proceed with the next iterations. Substituting Fourier mode $\sigma_m^n = \gamma^m e^{\phi h}$ in (16) gives

$$
\gamma = \frac{(\kappa_4 + \kappa_6) \cos(\phi h) + \kappa_5 + i(\kappa_6 - \kappa_4) \sin(\phi h)}{(\kappa_1 + \kappa_3) \cos(\phi h) + \kappa_2 + i(\kappa_3 - \kappa_1) \sin(\phi h)} = \frac{L_1 + iL_2}{L_3 + iL_4},
$$

where

$$
\begin{align*}
L_1 &= (\kappa_4 + \kappa_6) \cos(\phi h) + \kappa_5, \quad L_2 = (\kappa_6 - \kappa_4) \sin(\phi h), \\
L_3 &= \kappa_1 + \kappa_3 \cos(\phi h) + \kappa_2, \quad L_4 = (\kappa_3 - \kappa_1) \sin(\phi h),
\end{align*}
$$

For the magnitude of the growth factor $|\gamma| \leq 1$, the recurrence relation (15) based on the present scheme is unconditionally stable.
2.2 Quadractic B-spline approach (QBA)

The Quadratic B-spline basis functions \( B_m(x) \) at the knots \( x_m \) are defined as (P.M. Prenter 1975)

\[
B_m(x) = \frac{1}{h^2} \left\{ \begin{array}{ll}
(x_m+2 - x)^2 - 3(x_m+1 - x)^2 + 3(x_m - x)^2, & \text{if } x \in [x_{m-1}, x_m], \\
(x_m+2 - x)^2 - 3(x_m+1 - x)^2, & \text{if } x \in [x_m, x_{m+1}], \\
(x_m+2 - x)^2, & \text{if } x \in [x_{m+1}, x_{m+2}],
\end{array} \right.
\]

form the basis over the solution domain and each interval \([x_m, x_{m+1}]\) is covered by three successive quadratic B-spline functions. Identifying each finite element with the interval \([x_m, x_{m+1}]\) with nodes at \(x_m\) and \(x_{m+1}\) and using normalized variables, quadratic B-splines to be expressed in terms of local coordinates are:

\[
\psi_{m-1} = (1 - \eta)^2, \quad \psi_m = (1 + 2\eta - 2\eta^2), \quad \psi_{m+1} = \eta^2. \tag{20}
\]

Taking these quadratic B-spline basis functions, we have approximation \( U_N \) of the form

\[
U_N(\eta, \tau) = \sum_{j=m-1}^{m+1} \psi_j(\eta)(\sigma_j^n + \tau \Delta \sigma_j^n), \tag{21}
\]

where \( \psi_{m-1}, \psi_m, \psi_{m+1} \) act as shape functions for each element and \( \sigma^n_{m-1}, \sigma^n_m, \sigma^n_{m+1} \) are are nodal parameters with \( \Delta \sigma^n_{m-1}, \Delta \sigma^n_m, \Delta \sigma^n_{m+1} \) as the increments of the nodal parameters in a time step \( \Delta t \). Using equation (21) and quadratic B-spline functions (20), the values of \( U \) and \( U' \) at knots can be written in terms of parameter values as

\[
U_m = \sigma_{m-1} + \sigma_m, \quad U'_m = \frac{2}{\Delta x}(\sigma_m - \sigma_{m-1}). \tag{22}
\]

Using quadratic B-spline functions, our weight function is given by \( \delta \left( \frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 u}{\partial \eta^2} \right) = (\psi_j + \alpha \tau \psi_j' - \beta \psi_j'') \). Substituting in equation (7) yields the integral equation:

\[
\int_0^1 \int_0^1 \left( \frac{\partial u}{\partial \tau} + \alpha \frac{\partial u}{\partial \eta} - \beta \frac{\partial^2 u}{\partial \eta^2} \right) \left( \psi_i + \alpha \tau \psi_i' - \beta \psi_i'' \right) d\eta d\tau = 0. \tag{23}
\]

Inserting the local approximation (21) in equation (23), integrating the normalized term and collecting the terms containing \( \Delta \sigma_j \) and \( \sigma_j \), we obtain

\[
\sum_{j=m-1}^{m+1} \left\{ \int_0^1 \left( \psi_i \psi_j + \frac{\alpha}{2} (\psi_i \psi_j' + \psi_j \psi_i') + \left( \frac{\alpha^2}{3} + \beta \right) \psi_i' \psi_j' - \frac{\alpha \beta}{3} (\psi_i'' \psi_j + \psi_j'' \psi_i) + \frac{\beta^2}{3} \psi_i'' \psi_j'' \right) d\eta - \frac{\beta}{2} (\psi_i \psi_j' + \psi_j \psi_i') \right\} \Delta \sigma_j^n + \sum_{j=m-1}^{m+1} \left\{ \int_0^1 \left( \alpha \psi_i \psi_j' + \frac{\beta^2}{2} \psi_i'' \psi_j'' \right) d\eta - \beta (\psi_i \psi_j') \right\} \sigma_j^n = 0. \tag{24}
\]
Expanding for $i, j = m - 1, m, m + 1,$

\[
\left\{ \int_0^1 \psi_{m-1} \psi_{m-1} + \frac{1}{2} \left( \psi_{m-1} \psi_{m-1}' + \psi_{m-1}' \psi_{m-1} \right) + \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m-1} \psi_{m-1}' \right\} d\mu \\
- \frac{\beta}{2} \left( \psi_{m-1} \psi_{m-1}' \right) \left[ \int_0^1 \alpha \psi_{m-1} \psi_{m-1}' + \beta \psi_{m-1} \psi_{m-1}' + \frac{\alpha^2}{2} \psi_{m-1} \psi_{m-1}' \right] d\eta \\
+ \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m-1} \psi_{m-1}' \left[ \int_0^1 \alpha \psi_{m-1} \psi_{m-1}' + \beta \psi_{m-1} \psi_{m-1}' + \frac{\alpha^2}{2} \psi_{m-1} \psi_{m-1}' \right] d\eta \\
+ \frac{\alpha^2}{2} \left( \psi_{m-1} \psi_{m-1}' + \psi_{m-1} \psi_{m-1}' \right) + \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m-1} \psi_{m-1}' \right\} d\eta \\
- \frac{\beta}{2} \left( \psi_{m-1} \psi_{m-1}' \right) \left[ \int_0^1 \alpha \psi_{m-1} \psi_{m-1}' + \beta \psi_{m-1} \psi_{m-1}' + \frac{\alpha^2}{2} \psi_{m-1} \psi_{m-1}' \right] d\eta \\
+ \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m-1} \psi_{m-1}' \left[ \int_0^1 \alpha \psi_{m-1} \psi_{m-1}' + \beta \psi_{m-1} \psi_{m-1}' + \frac{\alpha^2}{2} \psi_{m-1} \psi_{m-1}' \right] d\eta \\
- \beta \left( \psi_{m-1} \psi_{m-1}' \right) \left[ \int_0^1 \alpha \psi_{m-1} \psi_{m-1}' + \beta \psi_{m-1} \psi_{m-1}' + \frac{\alpha^2}{2} \psi_{m-1} \psi_{m-1}' \right] d\eta \\
- \beta \left( \psi_{m-1} \psi_{m-1}' \right) \left[ \int_0^1 \alpha \psi_{m-1} \psi_{m-1}' + \beta \psi_{m-1} \psi_{m-1}' + \frac{\alpha^2}{2} \psi_{m-1} \psi_{m-1}' \right] d\eta = 0,
\]

\[
\left\{ \int_0^1 \psi_{m} \psi_{m} + \frac{1}{2} \left( \psi_{m} \psi_{m}' + \psi_{m}' \psi_{m} \right) + \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m} \psi_{m}' \right\} d\eta \\
- \frac{\beta}{2} \left( \psi_{m} \psi_{m}' \right) \left[ \int_0^1 \alpha \psi_{m} \psi_{m}' + \beta \psi_{m} \psi_{m}' + \frac{\alpha^2}{2} \psi_{m} \psi_{m}' \right] d\eta \\
+ \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m} \psi_{m}' \left[ \int_0^1 \alpha \psi_{m} \psi_{m}' + \beta \psi_{m} \psi_{m}' + \frac{\alpha^2}{2} \psi_{m} \psi_{m}' \right] d\eta \\
+ \frac{\alpha^2}{2} \left( \psi_{m} \psi_{m}' + \psi_{m} \psi_{m}' \right) + \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m} \psi_{m}' \right\} d\eta \\
- \frac{\beta}{2} \left( \psi_{m} \psi_{m}' \right) \left[ \int_0^1 \alpha \psi_{m} \psi_{m}' + \beta \psi_{m} \psi_{m}' + \frac{\alpha^2}{2} \psi_{m} \psi_{m}' \right] d\eta \\
+ \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m} \psi_{m}' \left[ \int_0^1 \alpha \psi_{m} \psi_{m}' + \beta \psi_{m} \psi_{m}' + \frac{\alpha^2}{2} \psi_{m} \psi_{m}' \right] d\eta \\
- \beta \left( \psi_{m} \psi_{m}' \right) \left[ \int_0^1 \alpha \psi_{m} \psi_{m}' + \beta \psi_{m} \psi_{m}' + \frac{\alpha^2}{2} \psi_{m} \psi_{m}' \right] d\eta \\
- \beta \left( \psi_{m} \psi_{m}' \right) \left[ \int_0^1 \alpha \psi_{m} \psi_{m}' + \beta \psi_{m} \psi_{m}' + \frac{\alpha^2}{2} \psi_{m} \psi_{m}' \right] d\eta = 0,
\]

and

\[
\left\{ \int_0^1 \psi_{m+1} \psi_{m-1} + \frac{1}{2} \left( \psi_{m+1} \psi_{m-1}' + \psi_{m-1}' \psi_{m+1} \right) + \left( \frac{\alpha^2}{2} + \frac{\beta}{2} \right) \psi_{m+1} \psi_{m-1}' \right\} d\eta
\]
The system (24) in the matrix form can be represented as

\[
\sum_{m=1}^{m+1} \left\{ X^e + \frac{\alpha}{2} (Y^e + (Y^e)^T) + \left( \frac{\alpha^2}{3} + \beta \right) Z^e - \frac{\alpha \beta}{3} (P^e + (P^e)^T) + \frac{\beta^2}{3} Q^e \right. \\
- \frac{\beta}{2} \left( \psi_{i,j} + \psi_{i,j}^e \right) d\eta \right\} \Delta \sigma_{n-1} + \left\{ \sum_{j=m+1}^{m+1} \alpha Y^e + \left( \frac{\alpha^2}{2} + \beta \right) Z^e - \frac{\alpha \beta}{2} \right. \\
\left. \times (P^e + (P^e)^T) + \frac{\beta^2}{2} Q^e - \beta(\psi_{i,j}^e) \right\} \sigma_{n-1} = 0,
\]

(25)

where \( i, j \) take values \( m-1, m, m+1 \) and \( m = 0, 1, ...N-1 \) and element matrices \( X^e, Y^e, Z^e, P^e, Q^e \) are given by

\[
X_{i,j}^e = \int_0^1 (\psi_{i,j}^e) d\eta, \quad Y_{i,j}^e = \int_0^1 (\psi_{i,j}^e) d\eta, \quad Z_{i,j}^e = \int_0^1 (\psi_{i,j}^e) d\eta \\
P_{i,j}^e = \int_0^1 (\psi_{i,j}^{e+1}) d\eta, \quad Q_{i,j}^e = \int_0^1 (\psi_{i,j}^{e+1}) d\eta.
\]

(26)

Taking contributions from all the elements and using \( \sigma = \sigma^n \) and \( \Delta \sigma = \sigma^{n+1} - \sigma^n \), the system of equations becomes

\[
\left\{ X + \frac{\alpha}{2} (Y + (Y)^T) + \left( \frac{\alpha^2}{3} + \beta \right) Z - \frac{\alpha \beta}{3} (P + (P)^T) + \frac{\beta^2}{3} Q \right. \\
- \frac{\beta}{2} \left( \psi_{i,j} + \psi_{i,j}^e \right) d\eta \right\} \sigma^{n+1} = \left\{ X + \frac{\alpha}{2} ((Y)^T - Y) - \frac{\alpha^2}{6} Z + \frac{\alpha \beta}{6} \right. \\
\left. \times (P + (P)^T) - \frac{\beta^2}{6} Q + \frac{\beta}{2} (\psi_{i,j}^e) \right\} \sigma^n = 0,
\]

(27)
A typical member of the matrix system (29) can be expressed in terms of nodal parameter $\sigma$ as:

\[
\begin{align*}
\kappa_1 & \sigma_{m-2}^{n+1} + \kappa_2 \sigma_{m-1}^{n+1} + \kappa_3 \sigma_m^{n+1} + \kappa_4 \sigma_{m+1}^{n+1} + \kappa_5 \sigma_{m+2}^{n+1} = \\
\kappa_6 & \sigma_{m-2}^n + \kappa_7 \sigma_{m-1}^n + \kappa_8 \sigma_m^n + \kappa_9 \sigma_{m+1}^n + \kappa_{10} \sigma_{m+2}^n,
\end{align*}
\]

(28)

where

\[
\begin{align*}
\kappa_1 &= \frac{1}{3} - \frac{2}{9} \alpha^2 - \frac{2}{3} \beta + \frac{4}{3} \beta^2, \\
\kappa_2 &= \frac{13}{15} - \frac{4}{9} \alpha^2 - \frac{4}{3} \beta - \frac{16}{3} \beta^2, \\
\kappa_3 &= \frac{33}{15} + \frac{4}{9} \alpha^2 + 4 \beta + 8 \beta^2, \\
\kappa_4 &= \frac{13}{15} - \frac{4}{9} \alpha^2 - \frac{4}{3} \beta - \frac{16}{3} \beta^2, \\
\kappa_5 &= \frac{13}{15} + \frac{5}{9} \alpha^2 + \frac{2}{3} \beta^2, \\
\kappa_6 &= \frac{1}{30} + \frac{a}{6} + \frac{\alpha^2}{9} - \frac{2}{3} \beta^2, \\
\kappa_7 &= \frac{13}{15} + \frac{5}{9} \alpha^2 + \frac{2}{3} \beta^2, \\
\kappa_8 &= \frac{33}{15} - \frac{2}{3} \alpha^2 - 4 \beta^2, \\
\kappa_9 &= \frac{13}{15} - \frac{5}{9} \alpha^2 + \frac{8}{3} \beta^2, \\
\kappa_{10} &= \frac{1}{30} - \frac{\alpha}{6} + \frac{\alpha^2}{9} - \frac{2}{3} \beta^2.
\end{align*}
\]

This system of equations can be solved iteratively after getting initial values from the given condition. Expression (27) in a more compact form can be expressed as

\[
[L] \{\Delta \sigma_j^n\} = [R] \{\sigma_j^n\},
\]

(29)

where $j = -1, 0, \ldots, N$ and $\{\Delta \sigma_j\} = \{\Delta \sigma_{-1}, \Delta \sigma_0, \Delta \sigma_1, \Delta \sigma_2, \ldots, \Delta \sigma_{N-4}, \Delta \sigma_{N-3}, \Delta \sigma_{N-2}, \Delta \sigma_{N-1}, \Delta \sigma_N\}^T$, $\{\sigma_j\} = \{\sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{N-4}, \sigma_{N-3}, \sigma_{N-2}, \sigma_N\}^T$. $L$ and $R$ are the assembled matrices of the form. Substituting the Fourier mode $\sigma_m^n = \gamma^n e^{i\varphi m h}$ in equation (28), we obtain

\[
\begin{align*}
\gamma^{n+1} e^{i m h \varphi} (\kappa_1 e^{-i h \varphi} + \kappa_2 e^{-i h \varphi} + \kappa_3 + \kappa_4 e^{i h \varphi} + \kappa_5 e^{2 i h \varphi}) &= \\
\gamma^n e^{i m h \varphi} (\kappa_6 e^{-2 i h \varphi} + \kappa_7 e^{-i h \varphi} + \kappa_8 + \kappa_9 e^{i h \varphi} + \kappa_{10} e^{2 i h \varphi}).
\end{align*}
\]

(30)

where $i = \sqrt{-1}$. Also, we have $\kappa_4 = \kappa_2$ and $\kappa_5 = \kappa_1$. So, equation (30) can be expressed as

\[
\begin{align*}
\gamma &= \frac{((\kappa_6 + \kappa_{10}) \cos(2 \varphi h) + (\kappa_7 + \kappa_9) \cos(\varphi h)) + \kappa_8}{(\kappa_1 + \kappa_5) \cos(2 \varphi h) + (\kappa_2 + \kappa_4) \cos(\varphi h) + \kappa_3} = \frac{Q_1 + i Q_2}{Q_3},
\end{align*}
\]

(31)

\[
\begin{align*}
Q_1 &= ((\kappa_6 + \kappa_{10}) \cos(2 \varphi h) + (\kappa_7 + \kappa_9) \cos(\varphi h)) + \kappa_8 \\
Q_2 &= ((\kappa_{10} - \kappa_6) \sin(2 \varphi h) + (\kappa_9 - \kappa_7) \cos(\varphi h)), \\
Q_3 &= (\kappa_1 + \kappa_5) \cos(2 \varphi h) + (\kappa_2 + \kappa_4) \cos(\varphi h) + \kappa_3,
\end{align*}
\]

For the magnitude of the growth factor $|\gamma| \leq 1$, above recurrence relation based on the present scheme is unconditionally stable.

3 TEST PROBLEMS

In this section we consider some test problems to illustrate the performance of the proposed scheme. Results are compared with the exact solution as well as with the other solutions available in the literature (I.A. Hassanien 2005; S.Kutluay 2004; Gülşu 2006; Dogan 2004; M. Gülşu 2005). In the first test problem, we have boundary conditions

\[ U(0, t) = 0, \quad U(1, t) = 0; \quad t > 0, \]

and initial condition \( U(x, 0) = \sin(\pi x), \quad 0 \leq x \leq 1, \) for Burgers equation (1). Analytical solution for this case is given by

\[ U(x, t) = 2\pi \nu \frac{\sum_{i=1}^{\infty} a_n \exp(-n^2\pi^2\nu t)n \sin(n\pi x)}{a_0 + \sum_{i=1}^{\infty} a_n \exp(-n^2\pi^2\nu t) \cos(n\pi x)}, \]

with the Fourier coefficients

\[ a_0 = \int_0^1 \exp\{-2\pi\nu\}^{-1}[1 - \cos(\pi x)] dx, \]
\[ a_n = 2 \int_0^1 \exp\{-2\pi\nu\}^{-1}[1 - \cos(\pi x)] \cos(n\pi x) dx, \quad n = 1, 2, 3... \]

For this problem, results obtained from the given scheme are documented in Table 1 for different values of \( \nu = 0.02, 0.1, 0.5. \) These computation are carried out for time step \( \Delta t = 0.01 \) and space step \( h = 0.005. \) Results obtained for different space steps at \( t = 0.1, \nu = 1.0 \) with time step \( \Delta t = 0.001 \) are presented and compared with (I.A. Hassanien 2005) in Table 2. From these tabular values, agreement between both numerical and exact values appears very satisfactory. Graphical representation of the results is shown through Figs. 1-6 with different values of \( \nu = 0.1, 0.7, 0.04 \) for linear B-spline approximation (LBA) and quadratic B-spline approximation (QBA). For the second case we have initial condition \( U_0 = 4x(1 - x) \) and the given homogeneous boundary conditions (32) for the Burgers equation (1). Analytical solution for this case is given by (33) where the Fourier coefficients are

\[ a_0 = \int_0^1 \exp(-x^2(3\nu))^{-1}(3 - 2x) dx, \]
\[ a_n = 2 \int_0^1 \exp(-x^2(3\nu))^{-1}(3 - 2x) \cos(n\pi x) dx, \quad n = 1, 2, 3, .... \]

Numerical solution of both the schemes together with corresponding exact solution is documented at some values over the domain of the problem in Table 3 and 4. Taking space step \( h = 0.0125 \) and time step \( \Delta t = 0.001, \) results are presented for \( \nu = 0.5, 0.2, 0.02 \) at different points in Table 3. In Table 4, comparison of results is made at \( t = 0.1 \) for \( \nu = 1.0, \Delta t = 0.001 \) for different mesh sizes. An additional comparison can also be seen between the present methods and method presented by (S.Kutluay 2004). A good agreement in numerical and exact values is evident through these Tables. Numerical results taking \( \nu = 1.0, 0.4, 0.03 \) at various time are graphed in Fig. 7-12 for the both the schemes. In the third test example, we have the Burgers equation (1) with boundary conditions (32) and initial condition at \( t = 1 \) with \( t_0 = \exp(1/8\nu), \) whose exact solution is given by

\[ U(x, t) = \frac{x/t}{1 + \sqrt{t/t_0} \exp\left(\frac{t}{4t_0}(x^2 - 1/4)\right)}, \quad t \geq 1, \quad 0 \leq x \leq 1. \]
Figure 1: Results obtained at $\nu = 0.1$ for first test problem using LBA.

Numerical solution of the algorithms together with the exact solution is documented in Table 5 for this case study. Results are visualized for $\nu = 0.01$ at $t = 1.0, 1.5, 2.0$ and for $\nu = 0.003$ at $t = 1.5, 2.0, 2.5$ at various times through Fig. 13-16.
Figure 2: Results obtained at $\nu = 0.1$ for first test problem using QBA.

Figure 3: Results obtained at $\nu = 0.7$ for first test problem using LBA.
Figure 4: Results obtained at $\nu = 0.7$ for first test problem using QBA.

Figure 5: Results obtained at $\nu = 0.04$ for first test problem using LBA.
Figure 6: Results obtained at \( \nu = 0.04 \) for first test problem using QBA.

Figure 7: Results obtained at \( \nu = 1.0 \) for second test problem using LBA.

Figure 8: Results obtained at $\nu = 1.0$ for second test problem using QBA.

Figure 9: Results obtained at $\nu = 0.4$ for second test problem using LBA.
Figure 10: Results obtained at $\nu = 0.4$ for second test problem using QBA.

Figure 11: Results obtained at $\nu = 0.03$ for second test problem using LBA.
Figure 12: Results obtained at $\nu = 0.03$ for second test problem using QBA.

Figure 13: Results obtained at $\nu = 0.01$ for third test problem using LBA.
Figure 14: Results obtained at $\nu = 0.01$ for third test problem using QBA.

Figure 15: Results obtained at $\nu = 0.003$ for third test problem using LBA.
Figure 16: Results obtained at $\nu = 0.003$ for third test problem using QBA.

4 CONCLUSION

In this paper, B-spline finite element method using linear and quadratic B-spline basis functions has been successfully used to develop the solution of Burgers’ equation. Different comparisons are made to check the accuracy of the numerical scheme. We have seen that the numerical technique presented here is capable enough of producing numerical solution of high accuracy. The B-spline FEM is very beneficial for getting the numerical solutions of the differential equations when continuity is the basic requirement. Given technique is flexible enough and can be applied to other complex problems which are difficult to solve directly.
REFERENCES


approximately burgers' equation using B-spline finite element method


Table 1: Comparison of results at different times for the first test problem at $\Delta t = 0.01$, $h = 0.005$.

<table>
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<th>QBA</th>
<th>Exact</th>
<th>LBA</th>
<th>QBA</th>
<th>Exact</th>
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Table 1: Comparison of results at different times for the first test problem at $\Delta t = 0.01$, $h = 0.005$. 

$x_n = 0.025$.
Table 2: Comparison of results for the first test problem at $t = 0.1, \nu = 1.0$ with $\Delta t = 0.001$.

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Table 3: Comparison of results at different times for the second test problem for $\Delta t = 0.001$, $h = 0.0125$.

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<td>0.008327</td>
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Table 4: Comparison of results for the second test problem at $t = 0.1$, $\nu = 1.0$ with $\Delta t = 0.001$.

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Table 5: Comparison of results at different times for the third test problem with $\nu = 0.1$, $\Delta t = 0.001$, $h = 0.001$.

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<th>QBA</th>
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<td>0.0758</td>
<td>0.0619</td>
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<td>0.0619</td>
</tr>
</tbody>
</table>

Table 5: Comparison of results at different times for the third test problem with $\nu = 0.1$, $\Delta t = 0.001$, $h = 0.001$. 