SOLUTION OF COUPLED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS USING HOMOTOPY PERTURBATION METHOD

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ABSTRACT

Aim of the paper is to investigate applications of Homotopy Perturbation Method (HPM) on non-linear physical problems. Some coupled non-linear partial differential equations are considered and solved numerically using Homotopy Perturbation Method. The results obtained by HPM are compared with those obtained by Adomian Decomposition Method. The behavior of Homotopic solution is shown through graphs. It is observed that HPM is an effective method with high accuracy.

Keywords: Coupled partial differential equation, Adomian decomposition method, Variational iteration method, Homotopy perturbation method

1 INTRODUCTION

Nonlinear evaluation equations arise in many areas of Science and Engineering, especially in mechanics, solid state physics, plasma physics, chemical physics etc. There is a large amount of literature available for obtaining numerical and explicit exact solutions of non-linear coupled differential equations. Many researchers have applied various methods to study the solutions of non-linear differential equations such as Darboux transformation (Matveev and Salle 1991), Hirota’s bilinear method (Hirota 1972), Tanh method (Baldwin et al. 2004; Wazwaz 2004), Finite difference method (Dehghan 2000) etc.

Recently, Variational iteration method (He 2000b; Wazwaz 2007; Golbabai and Javidi 2007; Tatari and Dehghan 2007) and Adomian decomposition method (Wazwaz 2000; El-Sayad et al. 2010) were used for solving non-linear problems. The Adomian decomposition method (Wazwaz 2000) is an efficient method for solving linear and nonlinear differential equations. This method is well suited to physical problems since it avoids the linearization, discretization and other restrictions.

He (He 1999; 2000a; 2003b; 2005; 2006) employed the fundamental idea of the homotopy in topology to propose a new method, namely, Homotopy Perturbation Method (HPM) for solving linear and nonlinear differential equations. The idea behind this method is introduction of a homotopy parameter, say $p$ which takes the values from 0 to 1. When $p = 0$, the system of equations usually reduces to a simplified form, which normally admit a rather simple solution. As $p$ gradually increases to 1, the system goes through a sequence of
deformation, the solution of each of which is close to that at the previous stage of deformation. Eventually at $p=1$, the system takes the original form of equation and final stage of deformation gives the desired result. One of the most remarkable features of the HPM is that usually only a few perturbation terms are sufficient to obtain a reasonably accurate solution.

The HPM has been employed to solve a large variety of linear and nonlinear problems. He (He 2006) applied HPM for solving non-linear boundary value problems. He (He 2003a) solved Blasius differential equation applying HPM. Ganji and Rafei (Ganji and Rafei 2006) studied solitary wave solutions for generalized non-linear Hirota–Satsuma coupled KdV partial differential equations. Biazer and Ghazvini (Biazer and Ghazvini 2009) presented a solution of systems of Volterra Integral equations of HPM. This approach was also applied to solve Lane-Emden type singular differential equations by HPM (Rafiq et al. 2009) and Abbabandy (Abbabandy 2007) employed He’s homotopy perturbation technique to solve functional integral equations.

The solutions obtained by the HPM show that the results of this technique are in excellent agreement with those of the Adomian decomposition method. A comparison between the HPM of He and the decomposition procedure of Adomian shows that the former is more effective than the latter, as the HPM can overcome the difficulties arising in calculating Adomian polynomials.

The homotopy perturbation method provides highly an accurate solution for non-linear problems in comparison with numerical techniques. It does not require large computer memory and discretization of variables. It can give the solutions for each point within the domain of interest, unlike the numerical solutions, which are available, for a particular run, only for a set of discrete points in the domain. The HPM avoids linearization and physically unrealistic assumptions.

Aim of the paper is to employ Homotopy Perturbation Method (HPM) to solve non-linear coupled differential equations and the results are compared with those obtained by Adomian decomposition method (Abdou and Wakil 2007).

2 BASIC IDEA OF HOMOTOPY PERTURBATION METHOD

Consider the following non-linear differential equation

$$A(w) - f(r) = 0, \quad r \in D,$$  \hspace{1cm} (1)

with the boundary conditions

$$B\left(w, \frac{\partial w}{\partial n}\right) = 0, \quad r \in \gamma,$$  \hspace{1cm} (2)

where $A$ is a general differential operator, $B$ is a boundary operator, $f(r)$ is known analytic function, $\gamma$ is the boundary of the domain $D$.
The operator A can generally be divided into two parts L and N, where L is linear, and N is nonlinear, therefore the equation (1) can be written as

\[ L(w) + N(w) - f(r) = 0 \]  \hspace{1cm} (3)

By using Homotopy technique, we construct a homotopy \( u'(r, p) : D \times [0,1] \rightarrow R \), which satisfies

\[ H(u', p) = (1-p)\left[ L(u') - L(w_0) \right] + p \left[ A(u') - f(r) \right] = 0, \]  \hspace{1cm} (4)

where \( p \in [0,1] \) is an embedding parameter and \( w_0 \) is the initial approximation of equation (1) which satisfies the boundary conditions.

Obviously, we get

\[ H(u', 0) = L(u') - L(w_0) = 0, \quad H(u', 1) = A(u') - f(r) = 0. \]  \hspace{1cm} (5)

The changing process of \( p \) from zero to unity is just that of \( u'(r, p) \) changing from \( w_0(r) \) to \( w(r) \). This is called deformation and also \( L(u') - L(w_0) \) and \( A(u') - f(r) \) are called homotopic in topology. If, the embedding parameter \( p \) (when \( 0 \leq x \leq 1 \)) is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of equation (4) can be given as power series in \( p \) as given below

\[ u = u_0' + pu_1' + p^2u_2' + \ldots, \]  \hspace{1cm} (6)

and setting \( p = 1 \) results in an approximate solution of equation (1) as

\[ u = \lim_{p \to 1} u = u_0' + u_1' + u_2' + \ldots. \]  \hspace{1cm} (7)

The series of equation (7) is convergent for most of the cases. However, the convergent rate depends on the nonlinear operator \( N(u') \), the following suggestions have already been made by He (1999):

(i) The second derivative of \( N(u') \) with respect to \( u \) must be small because the parameter may be relatively large i.e \( p \to 1 \), and

(ii) The norm of \( L^{-1} \left( \frac{\partial N}{\partial u} \right) \) must be smaller than one so that the series is convergent.
3 APPLICATIONS

Following examples illustrate the versatile nature of Homotopy Perturbation Method:

Example 1

Consider coupled system of nonlinear physical equations

\[ \frac{\partial u}{\partial t} = u(1-u-v) + \frac{\partial^2 u}{\partial x^2}, t > 0, \]  
(8)

\[ \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - uv, \]  
(9)

with the following initial conditions

\[ u(x,0) = \frac{e^{kx}}{(1 + e^{0.5kx})^2}, \]  
(10)

\[ v(x,0) = \frac{1}{(1 + e^{0.5kx})}, \]  
(11)

where \( k \) is constant.

To find the solution of equations (8) and (9) by HPM, we construct the following homotopy

\[ H(u', p) = (1-p)[u'_0 - u_0] + p[u'_1 - u'_0 - u' + u^2 + u'v] = 0, \]  
(12)

\[ H(v', p) = (1-p)[v'_0 - v_0] + p[v'_1 - v'_0 + u'v + u'v'] = 0, \]  
(13)

Where

\[ u' = u'_0 + pu'_1 + p^2u'_2 + p^3u'_3 + \ldots, \]  
(14)

\[ v' = v'_0 + pv'_1 + p^2v'_2 + p^3v'_3 + \ldots. \]  
(15)

Following equation (7), the variables \( u \) and \( v \) are obtained as given below

\[ u = \lim_{p \to 0} u' = u'_0 + u'_1 + u'_2 + u'_3 + \ldots, \]  
(16)

\[ v = \lim_{p \to 0} v' = v'_0 + v'_1 + v'_2 + v'_3 + \ldots. \]  
(17)
Substituting the equations (14) and (15) into the equations (12) and (13) and equating the like powers of \( p \), we get

\[
u_{tt} - \nu_{tt} = 0, \quad u_{tt} - u_{tt} - u_{tt}^2 + u_0 v_0 - u_{xx} = 0,
\]

\[
u_{tt} - u_{tt} - 2u_0 u_0' + u_0 v_0' + u_0 v_0' - u_{xx} = 0 \quad \text{and so on,} \quad (18)
\]

\[
u_{tt} - v_{tt} = 0, \quad v_{tt} + u_0 v_0' - v_{xx} = 0, \quad v_{tt} + u_0 v_0' + u_0 v_0' - v_{xx} = 0 \quad \text{and so on.} \quad (19)
\]

Let us choose \( u(x,0) = \frac{e^{kx}}{(1 + e^{0.5kx})} \) and \( v(x,0) = \frac{1}{(1 + e^{0.5kx})} \) as initial approximations, then from equations (18) and (19), various terms of \( u_t, u_{xx}, v_t \) and \( v_{xx} \) are obtained and given below.

\[
\begin{align*}
\nu_0(x,0) & = \frac{e^{kx}}{(1 + e^{0.5kx})^2}, \\
u_0'(x,0) & = \frac{1}{(1 + e^{0.5kx})^3}, \\
u_1 & = \frac{t \left(1+1.5k^2 \right) e^{1.5kx}}{(1 + e^{0.5kx})^4}, \\
u_2 & = \frac{\left\{ e^{0.5kx} \left[ (1+1.5k^2) e^{1.5kx} \right] + \left\{ (1 + e^{0.5kx}) e^{2kx} \right\} + \left\{ (1 + e^{0.5kx}) e^{3kx} \right\} - \left\{ 1 + (0.5k)^2 e^{2kx} \right\} \right\} t^2 / 2 \left(1 + e^{0.5kx} \right)^6, \\
u_2 & = \left\{ e^{0.5kx} \left[ -0.1875k^4 \right] e^{2kx} \left( 0.0625k^4 + k^2 \right) \right\} - \left\{ (0.5625k^4 + 2k^2 + k + 1) + e^{2kx} \left( 0.1875k^4 + k^2 + 1 \right) \right\} t^2 / 2 \left(1 + e^{0.5kx} \right)^5.
\end{align*}
\]

The numerical behaviors of approximate solutions of Homotopy Perturbation Method with different values of time are compared with those obtained by Adomian decomposition method and are shown through figures 1-4.
Figure 1: Variation of $u$ with respect to $x$ and $t$ applying HPM

Figure 2: Variation of $u$ with respect to $x$ and $t$ applying ADM (Abdou and Wakil 2007)
Figure 3: Variation of $v$ with respect to $x$ and $t$ applying HPM.

Figure 4: Variation of $v$ with respect to $x$ and $t$ applying ADM (Abdou and Wakil 2007)
Example 2

Consider the coupled system of nonlinear physical equations

\[
\frac{\partial u}{\partial t} = u(1-u^2-v) + \frac{\partial^2 u}{\partial x^2}, \quad t > 0,
\]

\[
\frac{\partial v}{\partial t} = v(1-u-v) + \frac{\partial^2 v}{\partial x^2},
\]

with the following initial conditions

\[
u(x,0) = \frac{e^{kx}}{1+e^{kx}},
\]

\[
v(x,0) = \frac{1+\frac{3}{4}(e^{kx})}{(1+e^{kx})^2},
\]

where \(k\) is constant.

To find solutions of equations (20) and (21) by HPM, we construct the following homotopy

\[
H(u', p) = (1-p)[u_i' - u_{0i}'] + p[u_i' - u_{0i}' - u_i + u_3 + u_i v_i] = 0,
\]

\[
H(v', p) = (1-p)[v_i' - v_{0i}'] + p[v_i' - v_{0i}' - v_i + v_i v_i + v_i^2] = 0,
\]

Where

\[
u' = u_0' + pu_1' + p^2 u_2' + p^3 u_3' + \ldots,
\]

\[
v' = v_0' + pv_1' + p^2 v_2' + p^3 v_3' + \ldots.
\]

Following equation (7), the variables \(u\) and \(v\) are obtained as given below

\[
u = \lim_{p \to 1} u' = u_0' + u_1' + u_2' + u_3' + \ldots.
\]

\[
v = \lim_{p \to 1} v' = v_0' + v_1' + v_2' + v_3' + \ldots.
\]

Substituting the equations (26) and (27) into the equations (24) and (25) and equating the like powers of \(p\), we get

\[
u_{0i}' - u_{0i}' = 0, \quad u_i' - u_{0x}' - u_0' + u_0 v_0 = 0, \quad u_{2i}' - u_i' + 3u_0^2 u_i' + u_0 v_i = 0, \quad u_i v_i - u_i' = 0 \text{ and so on,}
\]
\[ u_{tt} - v_{tt} = 0, \quad v_{tt} - v_{0xx} - v_0 = 0, \quad v_{2t} - v_i + u_0 v_i + u_i v_0 - v_{1xx} = 0 \quad \text{and so on} \]  

(31)

Let us choose \( u(x,0) = e^{kx} \) and \( v(x,0) = \frac{1+\frac{3}{4}(e^{kx})}{(1+e^{kx})^2} \) as initial approximations,

then from equations (30) and (31) various terms of \( u_i, u'_2 \) and \( v_i \) are obtained and given below

\[
u_0(x,0) = \frac{e^{(kx)}}{(1+e^{(kx)})}, \quad v_0(x,0) = \frac{1+\frac{3}{4}(e^{kx})}{(1+e^{kx})^2},
\]

\[
u_i = \left[ \frac{(k^2+5/4)e^{(2kx)}}{1+e^{(kx)}} \right] t, \quad v_i = \left[ \frac{(3+4k^2)e^{2kx}+(7-5k^2)e^{kx}+3k^2e^{3kx}+4}{4(1+e^{kx})^4} \right] t,
\]

\[
u_2 = \left[ \frac{(k^4+1+k^2)e^{3kx}+(-10k^4+9k^2-7/4)e^{(2kx)}+(11k^4-13k^2+13/16)e^{3kx}+(-k^4+3.75k^2-2.5)e^{(4kx)}}{1+e^{(kx)}^5} \right] t.
\]

The numerical behavior of approximate solutions of Homotopy Perturbation Method with different values of time are compared with those obtained by Adomian Decomposition Method (Adbou and Wakil 2007) and are shown through figures 5-8.

![Figure 5: Variation of \( u \) with to \( x \) and \( t \) applying HPM](image)
Figure 6: Variation of $u$ with respect to $x$ and $t$ applying ADM (Abdou and Wakil 2007)

Figure 7: Variation of $v$ with respect to $x$ and $t$ applying HPM.
Example 3

Tsunami is sea surface waves generated by large-scale underwater disturbances. Tsunami occurs due to earthquake-initiated seabed displacements, volcanic eruptions, landslides, impact of large objects such as meteors into the open ocean and underwater explosions etc. Such impulsive disturbances create water-wave motions where the entire water column from the bottom to the sea surface is set in motion. The surface waves generated by such motions have a very long wavelength compared with the depth of the ocean basin where they propagate. Consider the shallow-water model consisting of a pair of non-linear coupled partial differential equations

\[ u_t + uu_x + g \eta_x = 0, \]  
\[ \eta_t + (u(\eta + H))_x = 0, \]  

with the following initial conditions

\[ \eta(x,0) = h \sec h^2 \left( \frac{3h}{4d^3} x \right), \]  
\[ u(x,0) = \eta \sqrt{\frac{g}{d}}, \]  

\[ \text{(32)} \]  
\[ \text{(33)} \]  
\[ \text{(34)} \]  
\[ \text{(35)} \]  

**Figure 8:** Variation of \( \eta \) with respect to \( x \) and \( t \) applying ADM (Abdou and Wakil 2007)
where \( u \) is the tsunami velocity, \( \eta \) is surface elevation, \( H(x) \) is variable sea depth, \( g \) is acceleration due to gravity, \( x \) is the horizontal distance, \( t \) is the time, \( h = 2 \) (initial wave height), \( d = 20 \) (sea depth in the open ocean). Assuming shore slope = 0.2, that is \( H(x) = 0.2x - 20 \) (Gedi et al. 2005).

To find solutions of equations (32) and (33) by HPM, we construct the following homotopy

\[
H(u', p) = (1 - p)\left[u' - u_{0r}\right] + p\left[u' + u_{0r} + g\eta'_{0r}\right] = 0, \tag{36}
\]

\[
H(\eta', p) = (1 - p)\left[\eta' - \eta_{0r}\right] + p\left[\eta' + u\eta'_x + u\eta' + 0.2u + (0.2x - 20)u_x\right] = 0, \tag{37}
\]

Where

\[
u' = u_{0r} + pu_{1r} + p^2u_{2r} + p^3u_{3r} + \ldots, \tag{38}
\]

\[
\eta' = \eta_{0r} + p\eta_{1r} + p^2\eta_{2r} + p^3\eta_{3r} + \ldots. \tag{39}
\]

Following equation (7), the variables \( u \) and \( \eta \) are obtained as

\[
u = \lim_{p \to 0} u' = u_{0r} + u_{1r} + u_{2r} + u_{3r} + \ldots, \tag{40}
\]

\[
\eta = \lim_{p \to 0} \eta' = \eta_{0r} + \eta_{1r} + \eta_{2r} + \eta_{3r} + \ldots. \tag{41}
\]

Substituting the equations (38) and (39) into the equations (36) and (37) and equating the like powers of \( p \), we get

\[
u_{0r} - u_{0r} = 0, \quad \nu_{1r} + u_{0r}u_{0r} + g\eta_{0r} = 0, \quad \nu_{2r} + u_{0r}u_{1r} + u_{0r}u_{0r} + g\eta_{0r} = 0 \quad \text{and so on}, \tag{42}
\]

\[
\eta_{0r} - \eta_{0r} = 0, \tag{43}
\]

\[
\eta_{1r} + u_{0r}\eta_{0r} + 0.2u_{0r} + u_{0r}\eta_{0r} - (0.2x - 20)u_{0r} = 0, \tag{44}
\]

\[
\eta_{2r} + u_{0r}\eta_{1r} + 0.2u_{1r} + \eta_{0r}\eta_{1r} + u_{0r}\eta_{0r} = 0 \quad \text{and so on.} \tag{45}
\]

Let us choose \( u(x, 0) = \eta\sqrt{\frac{g}{d}} \) and \( \eta(x, 0) = h \sec h^2 \left( \sqrt{\frac{3h}{4d^2}}x \right) \), as initial approximation, then from equations (42) and (43) various terms of \( \nu', \eta'_1 \) and \( \eta'_2 \) are obtained and given below.

\[
\alpha = 0.01369306394
\]

\[
u_0 = 1.4\text{sech}(\alpha x)^2, \quad \nu_1 = \left[ 0.3794 \text{sech}(\alpha x)^2 \tanh(\alpha x) + 0.03794 \left( \text{sech}(\alpha x)^2 \tanh(\alpha) \right) \right]t,
\]

\( \eta_0 = 2 \sec h(\alpha x)^2, \)

\[ \eta_1 = \left[ \{0.153362305\text{sech}(\alpha x)^4 \tanh(\alpha x) - 2.8\text{sech}(\alpha x)^2\} ight. \]

\[ + \{0.038340576\text{sech}(\alpha x)^2 \tanh(\alpha x)\}^t \]

\[ \eta_2 = \left[ \{1.515738 \alpha \text{sech}(\alpha x)^6 \tanh(\alpha x)^2\} + \{0.7854888 \alpha \text{sech}(\alpha x)^8\} + \right. \]

\[ \left. \{7.84 \alpha - 0.028384 - 0.01073541 - 7.84 \alpha \text{sech}(\alpha x)^4 \tanh(\alpha x)\} \right] \]

\[ + \text{sech}(\alpha x)^2 \tanh(\alpha x)\{-0.284384 - (0.568768 \alpha \text{sech}(\alpha x)^6\} \]

\[ + \{0.568768(0.2x-20)0.1073534 \text{sech}(\alpha x)^4 \tanh(\alpha x)^2\} \]

\[ t^{3/2} \]

The numerical behavior of approximate solutions of Homotopy Perturbation Method with different values of time are compared with those obtained by Adomian Decomposition Method and are shown through figures 9-12.

![Figure 9: Variation of \( u \) with respect to \( x \) and \( t \) applying HPM](image-url)
Figure 10: Variation of $u$ with respect to $x$ and $t$ applying ADM

Figure 11: Variation of $\eta$ with respect to $x$ and $t$ applying HPM
4 CONCLUSIONS

The solution of some non-linear coupled partial differential equations are obtained by Homotopy Perturbation Method (HPM) and compared with earlier solutions obtained by Adomian Decomposition Method (ADM). To illustrate the feasibility and reliability of this method components of series are calculated for different non-linear partial differential equation model and evidently accuracy increases if more components are included in the series, but at the expense of considerable increase in the complexity of calculations.

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REFERENCES


