CHAOS AND HOPF BIFURCATION OF A FINANCE SYSTEM WITH DISTRIBUTED TIME DELAY

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ABSTRACT

A continuous time finance system with distributed time delay is firstly proposed and the complex dynamic behavior of the system is investigated in this paper. The system shows complex dynamics such as periodic, quasi-periodic, Hopf bifurcation, and chaotic behaviors, the theoretical results obtained in this paper are confirmed by numerical simulations. The model thus offers an explanation of the periodicity and irregularity in commercial markets. This research has important theoretical and practical meanings.

**Keywords**: Finance system, Distributed time delay, Chaotic behaviors, Hopf bifurcation, Numerical simulations.

1 INTRODUCTION

In the last few years, with the great development of nonlinear science, especially the Hopf bifurcation and chaotic theory, the investigation on the complex nonlinear system has recently become more and more prominent and many perfect results in nature and engineering field are yielded. In recent years, complex systems approach has been raised to an alternative scientific methodology to understand the highly complex dynamics of real financial and economic systems. There is growing interest in applying nonlinear dynamic to economic modeling, examples are the IS-LM model (Fanti and Manfredi 2007; Mihaela et al. 2007; Luigi and Mario 2005), the comprehensive national strength model (Xing 2007), the Kaldor-Kalecki business cycle model and other models proposed in various references (Wu and Wang 2010; Szydlowski 2005; Dumitru and Opris 2009). Utilizing the nonlinear dynamical theory to study the complexity of economy and finance systems has wide foreground, important theoretical and practical meaning, It is the developmental direction of complex nonlinear economic systems.

In complicated nonlinear systems, the finance system has been investigated in a vast amount of literature (Gao and Ma 2009; Chen 2008a; Chen 2008b) and their reference cited therein. It is well-known that time delay is an important factor of mathematical models in economy, Modern research on economic dynamics seems to be generating renewed interest in differential equations with time delay terms, This is because some economic phenomena can not be exhaustively described purely using differential equations. In a general, time delays have two cases: discrete delay and distributed time delay (continuous delay). For the models with discrete delay has

been investigated by many researchers (e.g. (Gao and Ma 2009; Chen 2008b) etc). Studies of the finance systems show that time delays create a wide variety of dynamic behaviors including Hopf bifurcation and chaotic behaviors, but few work devoted to the finance system with distributed time delay. In fact, the distributed time delay can more effectively describe the real economic system.

In this paper, we will develop a finance system with the distributed time delayed feedbacks. It is shown that the distributed time delayed feedbacks can be applied to control dynamical behaviors (e.g. period, quasi-periodicity, Hopf bifurcation and chaos) in the finance system. An outline of this paper is as follows: In section 2, we will presents a finance system recently reported in the literature, and introduces distributed time delay terms in it. In section 3, complicated dynamical behaviors are given by the Routh-Hurwitz theorem and numerical simulation, furthermore the permanence of the system is also discussed and the corresponding sufficient condition are given. In section 4, I present numerical simulation results and discuss the economical significance of the finance system. Finally, concluding comments are provided in sections 5.

2 MODEL FORMULATION AND PRELIMINARIES

The basic model we consider is based on the following finance system model and can be written as follows by our notations.

\[
\begin{align*}
\dot{x}(t) &= z(t) + (y(t) - a)x(t), \\
y(t) &= 1 - by(t) - x^2(t), \\
z(t) &= -x(t) - cz(t),
\end{align*}
\] (1)

where \(x(t), y(t)\) and \(z(t)\), respectively, represent the interest rate, the investment demand, the price index, \(a \geq 0\) represents the saving amount, \(b \geq 0\) is the cost per investment, and \(c \geq 0\) is the elasticity of demand of commercial markets. The system (1) is chaotic when \(a = 6, b = 0.1, c = 1\) (see Fig. 1, Fig. 2). The result is described by Gao and Ma (2009), where Fig. 1, Fig. 2 respectively, represent Phase portrait and t-X plane of system (1).

Over the last several years, there are extensive studies on the finance system with discrete delay (see, Gao and Ma (2009), Chen (2008b)). In the present paper, we add a distributed time delay (continuous delay) force \(k \int_{-\infty}^{t} F(t - \tau)y(\tau)d\tau\) to the second equation of the finance system (1). The function \(F(t)\) satisfies \(\int_{0}^{\infty} F(s)ds = 1\), that is the following delayed feedback control system:

\[
\begin{align*}
\dot{x}(t) &= z(t) + (y(t) - a)x(t), \\
y(t) &= 1 - by(t) - x^2(t) + k \int_{-\infty}^{t} F(t - \tau)y(\tau)d\tau, \\
z(t) &= -x(t) - cz(t),
\end{align*}
\] (2)

where \(\tau \geq 0\) is time delay and \(k\) indicates the strength of the feedback.

In order to investigate the effects of delay on the system, we let \(F(t) = de^{-dt}, d \geq 0\), and carry out the chain transform \(\psi(t) = \int_{-\infty}^{t} F(t - \tau)y(\tau)d\tau\).

Furthermore system (2) becomes:

Chaotic attractor

Figure 1: (Phase portrait), Typical dynamic behavior of system (1): $a = 4, b = 0.1, c = 1$. 

\[
\begin{align*}
\dot{x}(t) &= z(t) + (y(t) - a)x(t), \\
\dot{y}(t) &= 1 - by(t) - x^2(t) + k\psi(t), \\
\dot{z}(t) &= -x(t) - cz(t), \\
\dot{\psi}(t) &= d(y(t) - \psi(t))
\end{align*}
\]

(3)

3 ANALYSIS AND NUMERICAL SIMULATION OF THE MODEL

Lemma 1 when $b = k$, system (3) has two equilibria:

$P_1 = (1, a + \frac{1}{c}, -\frac{1}{c}, a + \frac{1}{c}), P_2 = (-1, a + \frac{1}{c}, \frac{1}{c}, a + \frac{1}{c})$.

In the following, we focus on the existence of local property at equilibria $P_1, P_2$ of the system(3). The Jacobian of system(3) at $P_1, P_2$ gives the characteristic polynomial:

\[
\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 = 0.
\]

(4)

Where

$\alpha_1 = d + c + b - \frac{1}{c}, \alpha_2 = dc + bc - \frac{b}{c} - \frac{d}{c}$.

Hence (4) becomes:

$\lambda^2(\lambda + d + b)(\lambda + c - \frac{1}{c}) = 0$.

Figure 2: (t-X plane), Typical dynamic behavior of system (1): $a = 4, b = 0.1, c = 1$.

Obviously, when $c = 1$, the eigenvalues corresponding to equilibria $P_1, P_2$ are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -(d + b), \lambda_4 = 0$. Then $P_1, P_2$ is a higher order equilibria.

Assume that the Lyapunov exponents of system(3) are $\lambda_i$ for $i = 1, 2, 3, 4$. Satisfying $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$.

Fig. 3 shows the Lyapunov exponents spectrum of system(3), then the dynamical behavior of system(3) can be classified as follows:

**Theorem 1**

(1) For $c > 1$, $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 < 0, \lambda_4 < 0$, system (3) has a 2-torus as shown in Fig. 4, Fig. 5.

(2) For $0.7 \leq c \leq 1, \lambda_1 > 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 < 0$, system (3) has a chaotic attractor as shown in Fig. 6, Fig. 7.

where (i) Fig. 4, Fig. 5 respectively, represent phase portrait and t-X plane of system (3), which has a 2-torus at $a = 6, b = 0.1, c = 11, k = 0.1, d = 5$. (ii) Fig. 6, Fig. 7 respectively, represent phase portrait and t-X plane of system (3), which has a chaotic attractor at $a = 6, b = 0.1, c = 0.7, k = 0.1, d = 5$.

**Proof**

From Fig. 3, and (Shaopu and Yongjun 2003), it is easy to see Lyapunov Exponents is one method of quantifying chaotic behavior. This set of average Lyapunov exponents is called the

Figure 3: Lyapunov exponents spectrum of system (3): $a = 6, b = 0.1, c = 2, k = 0.1, d = 5$.

spectrum of Lyapunov exponents. If at least one of the average Lyapunov exponents is positive, then we conclude that the system displays (on the average) divergence of nearby trajectories and is "truly" chaotic.

**Lemma 2**

When $b \neq k, c - b - abc + kac + k \leq 0$, system (3) has a unique equilibrium:

$$P_0(0, \frac{1}{b-k}, 0, \frac{1}{b-k}).$$

**Lemma 3**

When $b \neq k, c - b - abc + kac + k > 0$, system (3) has three equilibria:

$$P_0(0, \frac{1}{b-k}, 0, \frac{1}{b-k}).$$

$$P_1(\sqrt{\frac{c - b - abc + kac + k}{c}}, \frac{1 + ac}{c}, -\frac{1}{c} \sqrt{\frac{c - b - abc + kac + k}{c}}, \frac{1 + ac}{c}).$$

and

$$P_2(-\sqrt{\frac{c - b - abc + kac + k}{c}}, \frac{1 + ac}{c}, \sqrt{\frac{c - b - abc + kac + k}{c}}, \frac{1 + ac}{c}).$$

In this section, we investigate the hopf bifurcation near equilibrium $P_0(0, \frac{1}{b-k}, 0, \frac{1}{b-k})$ only. In what follows, we assume that the coefficients in system (3) satisfy the following condition:

$$c - b - abc + kac + k \leq 0.$$  \hspace{1cm} (H_1)

In the following we focus on the existence of local Hopf bifurcation at equilibrium $P_0(0, \frac{1}{b-k}, 0, \frac{1}{b-k})$ of the system (3). Let $x = x, y = y - \frac{1}{b-k}, z = z, \psi = \psi - \frac{1}{b-k}$, the linearization of system (3) at the equilibrium $P(0,0,0,0)$ can be written as:

$$
\begin{align*}
\dot{x}(t) &= z(t) + (\frac{1}{b-k} - a)x(t), \\
\dot{y}(t) &= -by(t) + k\psi(t), \\
\dot{z}(t) &= -x(t) - cz(t), \\
\dot{\psi}(t) &= d(y(t) - \psi(t))
\end{align*}
$$

(5)

The Jacobian of (4) evaluated at $P(0,0,0,0)$ gives the characteristic polynomial

$P(\lambda) = \lambda^4 + \beta_1 \lambda^3 + \beta_2 \lambda^2 + \beta_3 \lambda + \beta_4$

Coefficients of $P(\lambda)$:

$\beta_1 = b + d + \Phi, \quad \beta_2 = d(b - k) + \Phi(b + d) + \Psi, \quad \beta_3 = \Phi d(b - k) + \Psi(b + d), \quad \beta_4 = \Psi d(b - k)$. 

Where

$\Phi = a + c - \frac{1}{b-k}$. 

\[ \Psi = 1 - c\left(\frac{1}{b-k} - a\right). \]

When
\[ c - b - abc + kac + k \leq 0. \quad (H_1) \]

we have
\[ b - k \geq \frac{c}{ac+1} \geq 0. \]

Then we have
\[ \Phi = a + c - \frac{1}{b-k} \geq c - \frac{1}{c} \]
\[ \Psi = 1 - c\left(\frac{1}{b-k} - a\right) \geq 0. \]

Then, when \( c \geq 1 \), \( \Phi, \Psi \) are nonnegative.

Furthermore, all coefficients \( \beta_1, \beta_2, \beta_3, \beta_4 \) are strictly positive. So that no real positive and zero real eigenvalues exist, and thus the system can lose stability only because of a complex pair of eigenvalues crossing the imaginary axis. In order to rule out complex eigenvalues with positive real parts, we additionally need, from Routh-Hurwicz theorem:

\[ \Delta_1 = \beta_1 > 0, \Delta_2 = \beta_1\beta_2 - \beta_3 > 0, \]

\[ \Delta_3 = \beta_1 \beta_2 \beta_3 - \beta_2^2 - \beta_1^2 \beta_4 > 0, \]
\[ \Delta_4 = \beta_1 \beta_2 \beta_3 \beta_4 > 0. \]

We therefore only need to analyze the third condition, for with the equilibrium \( P(0,0,0,0) \) will be locally asymptotically stable for those parameter sets in correspondence of which the following inequality holds:

\[ \Delta_3 = d^2 (b - k)^2 + \Phi d (b - k) b 
+ \Phi d^2 (b - k) + \Phi^2 d (b - k) + \Psi^2 + b^2 \Psi 
+ 2 b d \Psi + d^2 \Psi + \Phi \Psi b + \Phi \Psi d - 2 d (b - k) \Psi > 0. \]  

(6)

Therefore, given the invariant positive signs of the coefficients of the characteristic equation, loses of stability can only occur by way of a Hopf bifurcation. Let us consider the bifurcation boundary at which \( \Delta_3 = 0 \). By choosing \( d \) as bifurcation parameter, the following proposition holds:

**Proposition 1**

The equilibrium \( P \) is unstable and a Hopf bifurcation can emerge only if \( d_0 \) is the border of the parameter \( d \).

where

\[ d_0 = -\frac{\gamma_2 + \sqrt{\gamma_2^2 - 4 \gamma_1 \gamma_3}}{2} \]
Figure 7: (t-X plane), (c1) a=6, b=0.1, c=0.7, k=0.1, d=5.

\[ \gamma_1 = (b - k)^2 + \Phi(b - k) + \Psi, \]
\[ \gamma_2 = \Phi(b - k)b + \Phi^2(b - k) + 2b\Psi + \Phi\Psi - 2(b - k)\Psi, \]
\[ \gamma_3 = \Psi^2 + b^2\Psi + \Phi\Psi b. \]

**Proof**

From (6), we know

\[ \gamma_1 d^2 + \gamma_2 d + \gamma_3 > 0. \] (7)

when \( k > 0 \), equation (7) permanently holds for arbitrary \( d \), hence, (6) is satisfied, then, the system (3) will stabilize to the equilibrium \( P_0 \), as shown in Fig. 8(d).

when

\[ k < 0, \gamma_2 < 0, \sqrt{\gamma_2^2 - 4\gamma_1\gamma_3} \geq 0 \]

we have

\[ d_0 = \frac{-\gamma_2 + \sqrt{\gamma_2^2 - 4\gamma_1\gamma_3}}{2} \]

If \( d = d_0 \), Hopf bifurcation will occur.

From lemmas 2, lemmas 3, and proposition 1, it is easy to obtain the following theorem.

**Theorem 2**

suppose that the conditions \( b \neq k \), and \( H_1 \) are satisfied.

(i) If \( k > 0 \), the system (3) will stabilize to the equilibrium \( P_0 \).

(ii) If \( k < 0, \gamma_2 < 0, \sqrt{\gamma_2^2 - 4\gamma_1 \gamma_3} \geq 0 \), when \( d > d_0 \), then the equilibrium \( P_0 \) of system (3) is stable and unstable when \( 0 < d < d_0 \); system (3) can undergo a Hopf bifurcation at \( d = d_0 \).

**Proof**

(i) when \( k > 0 \), (6) permanently holds, hence, from Routh-Hurwicz theorem, system (3) is stable.

(ii) From lemmas 2, lemmas 3, proposition 1, and conditions \( k < 0, \gamma_2 < 0, \sqrt{\gamma_2^2 - 4\gamma_1 \gamma_3} \geq 0 \), it is easy to see, when \( d > d_0 \), (4) is satisfied, hence, from Routh-Hurwicz theorem, system (3) is stable; therefore, when, \( 0 < d < d_0 \), (4) is not satisfied. Therefore, given the invariant positive signs of the coefficients of the characteristic equation, loses of stability can only occur by way of a Hopf bifurcation, \( d = d_0 \) is the critical value which equation (4) can be tenable, hence when \( d = d_0 \), system (3) can undergo a Hopf bifurcation.
NUMERICAL ANALYSIS AND DISCUSSION

Our focus so far has been on the dynamic analysis of system (1) and (3). Now we study how the control parameters affects the dynamic behavior of the systems by numerical simulations.

Let $a = 6, b = 0.1, c = 1$, then system (1) exists chaotic attractor. This can be seen clearly from Fig. 1, Fig. 2.

Then from section two, when $a = 6, b = 0.1, c = 1, k = 0.1, d = 5$. we know that system (3) exists a 2-torus as shown in Fig. 4, Fig. 5.

When $a = 6, b = 0.1, c = 0.7, k = 0.1, d = 5$, system (3) has a chaotic attractor as shown in Fig. 6, Fig. 7.

when $a = 1, b = 2, c = 1, k = 1, d = 2$ then equilibrium $P_0$ is stable, this can be seen clearly from Fig. 8(d).

For above parameters, let $a = 1.87, b = 0.05, c = 1, k = -0.3$, from theorem 2, we have $d_0 = 0.005445$, we know that finance system (3) undergo a Hopf bifurcation at at $d_0$. When $d < d_0$, we take $a = 1.87, b = 0.05, c = 1, k = -0.3, d = 0.004$, system (3) will outset chaos. This is shown in Fig. 9, Fig. 10.

From the above theoretical analysis and numerical simulation, we can see that the finance system (3) with distributed time delay such a wide range of possesses complex dynamic behaviors including equilibria, 2-tori, 3-tori, Hopf bifurcation and strange attractors. This observation...
indicates that the distributed delay is a sensitive factor for the finance system, and Hopf bifurcation, chaos can be suppressed by choosing proper parameters.

When the cost per investment $b$ is equal to the strength of the feedback $k$, the equilibria $P_1, P_2$ will occur 2-tori, ultimately chaotic attractor, when parameters $a, b, c, d, k$ take proper parameters. Obviously, with the decreasing of the elasticity of demand of commercial markets, system (3) will occurs chaotic attractor, the system (3) will lose energy, which is coincident with the practice instance. When $b \neq k$, form numerical simulation we see system (3) will stabilizes to equilibrium $P_0$, but at this time the interest rate and the price index are all zero; this is unpractical, with the changing of $k$, system (3) state will soon change from a stable equilibrium to unstable, and can occurs Hopf bifurcation, can outset chaos.

A key result we obtain shows how an economy’s developments depends upon the available of its economy parameters, so if the parameters will be kept a proper level, then economy may develop quickly and healthily.

5 CONCLUSION

We studied the finance system with the distributed time delay. This paper indicates that the complex dynamic behavior in such an economic system can be controlled under appropriate feedback strengths and delay times, the feedbacks either suppress or enhance the dynamic behavior. This paper also show that the complex dynamic behavior of transitions is a fundamental feature of nonlinear finance system. Mathematical modeling of the finance system can deepen

our understanding of sudden major changes of economic variables often encountered in economic system.

The research of paper is a new breakthrough about the controlling of nonlinear finance system, and a new investigation about the stability and Hopf bifurcation on this finance system; it has important theoretical and practical meanings.

REFERENCES


