

QUASI-STATIC THERMAL STRESSES IN A THICK CIRCULAR PLATE DUE TO AXISYMMETRIC HEAT SUPPLY

V.S. Kulkarni¹ and K.C. Deshmukh²

¹*Department of Mathematics, Govt. College of Engineering, Aurangabad – 431 005, Maharashtra, INDIA*

Email: vinayakskulkarni1@rediffmail.com

²*Post-Graduate Department of Mathematics, Nagpur University, Nagpur – 440 010, Maharashtra, INDIA*

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ABSTRACT

The present paper deals with the determination of transient thermal stresses in a thick circular plate of different metals subjected to arbitrary heat flux on the upper and lower surfaces where as the fixed circular edge is thermally insulated and initial temperature of a thick plate is kept at zero. The governing heat conduction equation has been solved by using Laplace transform technique. The results are obtained in a series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and illustrated graphically. The numerical results are compared for different metal plates.

Keywords: Transient, thermoelastic problem, axisymmetric, thermal stresses.

1 INTRODUCTION

During the second half of the twentieth century, nonisothermal problems of the theory of elasticity became increasingly important. This is due to their wide application in diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of the aircraft structure.

Ishihara *et al.* (Ishihara *et al.* 1997) discussed transient thermoelastoplastic bending problem making use of strain increment theorem and determine temperature and thermoelastoplastic deformation for the heating and cooling processes in a thin circular plate subjected to partially distributed and axisymmetric heat supply on the upper face with the help of the generalized integral transforms. Hany *et al.* (Hany *et al.* 1994) solved the two-dimensional problem of a thick plate whose lower and upper surfaces are traction free and subjected to a given axisymmetric temperature distribution was considered within the context of the theory of generalized thermoelasticity with one relaxation time. Potential functions together with Laplace and Hankel transform techniques are used to derive the solution in the transformed domain. Rokne *et al.* (Rokne *et al.* 2004) found the solution of an axisymmetric Boussinesq type problem for half space under optimal heating of arbitrary profile. Sharma *et al.* (Sharma *et al.* 2004) studied the behavior of thermoelastic axisymmetric thick plate under lateral loads

and obtained the results for radial and axial displacements and temperature change have been computed numerically and illustrated graphically for different theories of generalized thermoelasticity. Yoshihiro Ochiai (Ochiai 1996) shows that the axisymmetric problem of steady thermoelasticity with nonuniform heat generation over the region can be easily solved without a domain integral by means of BEM. This method can also be applied to steady thermal stress problems under complicated general heat generation. However, for general heat generation the domain must be divided into small areas in which distributions of the heat generation approximately satisfy the Laplace equation. Kulkarni *et al.* (Kulkarni *et al.* 2007) determined displacement and thermal stresses in a thick circular plate due to arbitrary heat supply on the upper surface where as lower surface is at zero temperature and fixed circular edge is thermally insulated.

Recently Noda *et al.* (Noda *et al.* 2003) considered infinite thick circular plate and discuss thermal stresses due to arbitrary heat flux on the upper and lower surfaces, while we consider finite thick circular plates of different metals and discussed thermal stresses. The numerical results are compared for different metal plates. Due to arbitrary heat flux on the upper and lower surfaces, plates expand in axial direction and bend concavely at the center. This expansion and bending is in the proportion of the thermal diffusivity of metal. Which is new and novel contribution of this paper. The results presented here are more useful in engineering problem particularly in the determination of the state of strain in thick circular plate constituting foundation of containers for hot gases or liquids, in the foundation for furnaces etc.

2 FORMULATION OF THE PROBLEM

Consider a thick circular plate of radius a and thickness $2b$ defined by $0 \leq r \leq a, -b \leq z \leq b$. Let the plate be subjected to a transient axisymmetric temperature field on the radial and axial directions of the cylindrical coordinate system. Initially the plate is kept at zero temperature.

The arbitrary heat flux $\frac{Qf(r)}{\lambda}$ is prescribed over the upper surface ($z = b$) and the lower surface ($z = -b$). The fixed circular edge ($r = a$) thermally insulated. Under these more realistic prescribed conditions, the transient thermal stresses are required to be determined.

The differential equation governing the displacement potential function $\phi(r, z, t)$ is given in Noda *et al.* (Noda *et al.* 2003).

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = K \tau \tag{1}$$

where K is the restraint coefficient and temperature change $\tau = T - T_i$. T_i is initial temperature. Displacement function ϕ is known as Goodier’s thermoelastic potential.

The temperature of the plate at time t satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \frac{\partial T}{\partial t} \tag{2}$$

with the conditions

$$\frac{\partial T}{\partial z} = \pm \frac{Qf(r)}{\lambda} \quad \text{for } z = \pm b, \quad 0 \leq r \leq a, \quad \text{for all time } t \quad (3)$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = a, \quad -b \leq z \leq b \quad (4)$$

and

$$T = 0 \quad \text{at } t = 0 \quad (5)$$

where k is the thermal diffusivity of the material of the plate.

The displacement function in the cylindrical coordinate system are represented by the Michell's function as defined in Noda *et al.* (Noda *et al.* 2003) as

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \quad (6)$$

and

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}. \quad (7)$$

The Michell's function M must satisfy

$$\nabla^2 \nabla^2 M = 0, \quad (8)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (9)$$

The component of the stresses are represented by the thermoelastic displacement potential ϕ and Michell's function M as

$$\sigma_{rr} = 2G \left[\frac{\partial^2 \phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z} \left(\nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2} \right) \right] \quad (10)$$

$$\sigma_{\theta\theta} = 2G \left[\frac{1}{r} \frac{\partial \phi}{\partial r} - K\tau + \frac{\partial}{\partial z} \left(\nu \nabla^2 M - \frac{1}{r} \frac{\partial M}{\partial r} \right) \right] \quad (11)$$

$$\sigma_{zz} = 2G \left[\frac{\partial^2 \phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z} \left((2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \quad (12)$$

and

$$\sigma_{rz} = 2G \left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left((1-\nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \tag{13}$$

where G and ν are the shear modulus and Poisson’s ratio respectively.

For traction free surfaces stress functions

$$\sigma_{zz} = \sigma_{rz} = 0 \text{ at } z = \pm b \text{ for the thick plate.} \tag{14}$$

Equations (1) to (14) constitute mathematical formulation of the problem.

3 THE SOLUTION

Taking Laplace transform of equations (2) to (4) w.r.t. t and using equation (5), one obtains

$$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{T}}{\partial r} + \frac{\partial^2 \bar{T}}{\partial z^2} = \frac{P}{k} \bar{T} \tag{15}$$

with boundary conditions

$$\frac{\partial \bar{T}}{\partial z} = \pm \frac{Qf(r)}{\lambda P} \text{ at } z = \pm b \tag{16}$$

$$\frac{\partial \bar{T}}{\partial r} = 0 \text{ at } r = a \tag{17}$$

where P is Laplace transform parameter.

Now applying method of separation of variables to solve equation (15), one obtains

$$\bar{T} = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) \cosh[(\sqrt{\alpha_m^2 + q^2})Z] \tag{18}$$

where $q^2 = \frac{P}{K}$ and α_m is m the positive root of transcendental equation

$$J_1(\alpha \cdot a) = 0 \tag{19}$$

Now constant A_m can be obtained by using equations (16) and (18)

$$A_m = \frac{2Q f(S)}{a^2 \lambda P \sqrt{\alpha_m^2 + q^2} J_0^2(\alpha_m a) \sinh[(\sqrt{\alpha_m^2 + q^2})b]} \tag{20}$$

where

$$f(S) = \int_0^a r J_0(\alpha_m r) f(r) dr \quad (21)$$

Using equation (20) in equation (18) one obtains

$$\bar{T} = \sum_{m=1}^{\infty} \left[\frac{2Q f(S) J_0(\alpha_m r) \cosh[(\sqrt{\alpha_m^2 + q^2})z]}{a^2 \lambda P \sqrt{\alpha_m^2 + q^2} J_0^2(\alpha_m a) \sinh[(\sqrt{\alpha_m^2 + q^2})b]} \right] \quad (22)$$

Taking inverse Laplace transform of equation (22) one obtains the expression for temperature,

$$T = \left(\frac{2Q}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left\{ \frac{f(S) J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \left[\frac{\cosh(\alpha_m z)}{\alpha_m \sinh(\alpha_m b)} - \frac{e^{-\alpha_m^2 kt}}{\alpha_m^2 b} \right. \right. \\ \left. \left. + 2b \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(\frac{n\pi z}{b}\right) e^{-k \left[(n^2 \pi^2 + \alpha_m^2 b^2) \frac{t}{b^2} \right]}}{(n^2 \pi^2 + \alpha_m^2 b^2)} \right] \right\} \quad (23)$$

$$\text{Since initial temperature } T_i = 0, \tau = T. \quad (24)$$

Now let's assume Michell's function M , which satisfies condition (8) as,

$$M = \left(\frac{2QK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{f(S) J_0(\alpha_m r)}{J_0^2(\alpha_m a)} [B_{mn} \sinh(\alpha_m z) + C_{mn} \alpha_m z \cosh(\alpha_m z)] \right\} \quad (25)$$

where B_{mn} and C_{mn} are the arbitrary functions, which can be determined finally by using condition (14)

To obtain displacement potential ϕ using equations (23) and (24) in equation (1) one have,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{2QK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left\{ \frac{f(S) J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \left[\frac{\cosh(\alpha_m z)}{\alpha_m \sinh(\alpha_m b)} - \frac{e^{-\alpha_m^2 kt}}{\alpha_m^2 b} \right. \right. \\ \left. \left. + 2b \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(\frac{n\pi z}{b}\right) e^{-k \left[(n^2 \pi^2 + \alpha_m^2 b^2) \frac{t}{b^2} \right]}}{(n^2 \pi^2 + \alpha_m^2 b^2)} \right] \right\} \quad (26)$$

Considering first term of equation (26) as

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_1}{\partial r} + \frac{\partial^2 \phi_1}{\partial z^2} = \left(\frac{2QK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left[\frac{f(S) J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \times \frac{\cosh(\alpha_m z)}{\alpha_m \sinh(\alpha_m b)} \right] \tag{27}$$

To solve equation (27) assume

$$\phi_1 = \sum_{m=1}^{\infty} [D_m J_0(\alpha_m r) z \sinh(\alpha_m z)] \tag{28}$$

Using equation (28) in equation (27) one obtains

$$D_m = \frac{QKf(S)}{a^2 \lambda \alpha_m^2 J_0^2(\alpha_m a) \sinh(\alpha_m b)}$$

Hence

$$\phi_1 = \left(\frac{QK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left[\frac{f(S) J_0(\alpha_m r) Z \sinh(\alpha_m z)}{\alpha_m^2 J_0^2(\alpha_m a) \sinh(\alpha_m b)} \right] \tag{29}$$

Now considering second and third term of equation (26) as

$$\frac{\partial^2 \phi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_2}{\partial r} + \frac{\partial^2 \phi_2}{\partial z^2} = \left(\frac{2QK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left\{ \frac{f(S) J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \left[\frac{-e^{-\alpha_m^2 kt}}{\alpha_m^2 b} + 2b \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos\left(\frac{n\pi z}{b}\right) e^{-k \left[(n^2 \pi^2 + \alpha_m^2 b^2) \frac{t}{b^2} \right]}}{(n^2 \pi^2 + \alpha_m^2 b^2)} \right] \right\} \tag{30}$$

To solve equation (30) using

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \approx \frac{1}{k} \frac{\partial}{\partial t} \quad \text{in equation (30)(31)}$$

and on integrating w.r.t. t , one obtains

$$\phi_2 = \left(\frac{2QK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left\{ \frac{f(s) J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \left[\frac{e^{-\alpha_m^2 kt}}{\alpha_m^4 b} + 2b^3 \sum_{n=1}^{\infty} \frac{(-1)^n \cos\left(\frac{n\pi z}{b}\right) e^{-k \left[(n^2 \pi^2 + \alpha_m^2 b^2) \frac{t}{b^2} \right]}}{(n^2 \pi^2 + \alpha_m^2 b^2)^2} \right] \right\} \tag{31}$$

Finally $\phi = \phi_1 + \phi_2$

$$\phi = \left(\frac{2QK}{a^2\lambda} \right) \sum_{m=1}^{\infty} \left\{ \frac{f(s)J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \left[\frac{z \sinh(\alpha_m z)}{2\alpha_m^2 \sinh(\alpha_m b)} + \frac{e^{-\alpha_m^2 kt}}{\alpha_m^4 b} \right. \right. \\ \left. \left. + 2b^3 \sum_{n=1}^{\infty} \frac{(-1)^n \cos\left(\frac{n\pi z}{b}\right) e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)t}{b^2}\right]}}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \right] \right\} \quad (32)$$

Now using equations (23), (24), (25), and (32) in equations (6), (7), and (10) to (13), one obtains the expressions for displacement and stresses as

$$u_r = \left(\frac{2QK}{a^2\lambda} \right) \sum_{m=1}^{\infty} \left\{ \left[\frac{f(s)J_1(\alpha_m r)}{J_0^2(\alpha_m a)} \right] \left[\frac{-z \sinh(\alpha_m z)}{2\alpha_m \sinh(\alpha_m b)} - \frac{e^{-\alpha_m^2 kt}}{\alpha_m^3 b} \right. \right. \\ \left. \left. - 2\alpha_m b^3 \sum_{n=1}^{\infty} \frac{(-1)^n \cos\left(\frac{n\pi z}{b}\right) e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)t}{b^2}\right]}}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \right] \right. \\ \left. + \alpha_m^2 B_{mn} \cosh(\alpha_m z) + \alpha_m^2 C_{mn} [\cosh(\alpha_m z) + \alpha_m z \sinh(\alpha_m z)] \right\} \quad (33)$$

$$u_z = \left(\frac{2QK}{a^2\lambda} \right) \sum_{m=1}^{\infty} \left\{ \left[\frac{f(s)J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \right] \left[\frac{\sinh(\alpha_m z) + \alpha_m z \cosh(\alpha_m z)}{2\alpha_m^2 \sinh(\alpha_m b)} \right. \right. \\ \left. \left. - 2b^2 \pi \sum_{n=1}^{\infty} \frac{(-1)^n n \sin\left(\frac{n\pi z}{b}\right) e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)t}{b^2}\right]}}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \right] \right. \\ \left. - \alpha_m^2 B_{mn} \sinh(\alpha_m z) + \alpha_m^2 C_{mn} [2(1-2\nu) \sinh(\alpha_m z) - \alpha_m z \cosh(\alpha_m z)] \right\} \quad (34)$$

$$\sigma_{rr} = \left(\frac{4GQK}{a^2\lambda} \right) \sum_{m=1}^{\infty} \left\{ \left[\frac{f(s)}{J_0^2(\alpha_m a)} \right] \left[\left(\frac{J_1(\alpha_m r)}{r} - \alpha_m J_0(\alpha_m r) \right) \frac{z \sinh(\alpha_m z)}{2\alpha_m \sinh(\alpha_m b)} \right. \right. \\ \left. \left. - \frac{J_0(\alpha_m r) \cosh(\alpha_m z)}{\alpha_m \sinh(\alpha_m b)} + \left[\frac{J_1(\alpha_m r)}{r} \right] \left[\frac{e^{-\alpha_m^2 kt}}{\alpha_m^3 b} \right] \right. \right. \\ \left. \left. + 2b \sum_{n=1}^{\infty} \frac{(-1)^n \cos\left(\frac{n\pi z}{b}\right)}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \left[\frac{\alpha_m b^2 J_1(\alpha_m r)}{r} + n^2 \pi^2 J_0(\alpha_m r) \right] \left[e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)t}{b^2}\right]} \right] \right. \right. \\ \left. \left. + \alpha_m^2 B_{mn} \left[\alpha_m J_0(\alpha_m r) - \frac{J_1(\alpha_m r)}{r} \right] \cosh(\alpha_m z) \right. \right. \\ \left. \left. + \alpha_m^2 C_{mn} \left[2\nu \alpha_m J_0(\alpha_m r) \cosh(\alpha_m z) \right. \right. \right. \\ \left. \left. \left. + \left(\alpha_m J_0(\alpha_m r) - \frac{J_1(\alpha_m r)}{r} \right) (\cosh(\alpha_m z) + \alpha_m z \sinh(\alpha_m z)) \right] \right\} \quad (35)$$

$$\begin{aligned} \sigma_{\theta\theta} = & \left(\frac{4GQK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left\{ \left[\frac{f(s)}{J_0^2(\alpha_m a)} \right] \left[-J_0(\alpha_m r) \frac{\cosh(\alpha_m z)}{\alpha_m \sinh(\alpha_m b)} - \left[\frac{J_1(\alpha_m r)}{r} \right] \frac{z \sinh(\alpha_m z)}{2\alpha_m \sinh(\alpha_m b)} \right. \right. \\ & + \left[\alpha_m J_0(\alpha_m r) - \frac{J_1(\alpha_m r)}{r} \right] \frac{e^{-\alpha_m^2 kt}}{\alpha_m^3 b} + 2b \sum_{n=1}^{\infty} \frac{(-1)^n \cos\left(\frac{n\pi z}{b}\right) e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)}{b^2}t\right]}}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \\ & \times \left[(n^2\pi^2 + \alpha_m^2 b^2)J_0(\alpha_m r) - \alpha_m b^2 \frac{J_1(\alpha_m r)}{r} \right] + \alpha_m^2 B_{mn} \left[\frac{J_1(\alpha_m r)}{r} \right] \cosh(\alpha_m z) \\ & \left. \left. + \alpha_m^2 C_{mn} \left[\frac{2\nu\alpha_m J_0(\alpha_m r) \cosh(\alpha_m z)}{+ \frac{J_1(\alpha_m r)}{r} (\cosh(\alpha_m z) + \alpha_m z \sinh(\alpha_m z))} \right] \right\} \right\} \end{aligned} \tag{36}$$

$$\begin{aligned} \sigma_{zz} = & \left(\frac{4GQK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left\{ \left[\frac{f(s)J_0(\alpha_m r)}{J_0^2(\alpha_m a)} \right] \left[\frac{z \sinh(\alpha_m z)}{2 \sinh(\alpha_m b)} + \frac{e^{-\alpha_m^2 kt}}{\alpha_m^2 b} \right. \right. \\ & + 2\alpha_m^2 b^3 \sum_{n=1}^{\infty} \frac{(-1)^n \cos\left(\frac{n\pi z}{b}\right) e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)}{b^2}t\right]}}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \\ & \left. \left. - \alpha_m^3 B_{mn} \cosh(\alpha_m z) + \alpha_m^3 C_{mn} [(1 - 2\nu) \cosh(\alpha_m z) - \alpha_m z \sinh(\alpha_m z)] \right] \right\} \end{aligned} \tag{37}$$

$$\begin{aligned} \sigma_{rz} = & \left(\frac{4GQK}{a^2 \lambda} \right) \sum_{m=1}^{\infty} \left\{ \left[\frac{f(s)J_1(\alpha_m r)}{J_0^2(\alpha_m a)} \right] \left[\frac{-\sinh(\alpha_m z) - \alpha_m z \cosh(\alpha_m z)}{2\alpha_m \sinh(\alpha_m b)} \right. \right. \\ & + 2b^2 \pi \alpha_m \sum_{n=1}^{\infty} \frac{(-1)^n n \sin\left(\frac{n\pi z}{b}\right) e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)}{b^2}t\right]}}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \\ & \left. \left. + \alpha_m^3 B_{mn} \sinh(\alpha_m z) + \alpha_m^3 C_{mn} [2\nu \sinh(\alpha_m z) + \alpha_m z \cosh(\alpha_m z)] \right] \right\} \end{aligned} \tag{38}$$

In order to satisfy condition (14), solving equations (37) and (38) for B_{mn} and C_{mn} one obtains

$$\begin{aligned} B_{mn} = & \sum_{m=1}^{\infty} \left\{ \frac{1 - 2\nu}{2\alpha_m^4 \sinh(\alpha_m b)} + \frac{2\nu \sinh(\alpha_m b) + \alpha_m b \cosh(\alpha_m b)}{\alpha_m^4 [\sinh(\alpha_m b) \cosh(\alpha_m b) + \alpha_m b]} \right. \\ & \left. \times \left[\frac{e^{-\alpha_m^2 kt}}{\alpha_m b} + 2\alpha_m^3 b^3 \sum_{n=1}^{\infty} \frac{e^{-k\left[\frac{(n^2\pi^2 + \alpha_m^2 b^2)}{b^2}t\right]}}{(n^2\pi^2 + \alpha_m^2 b^2)^2} \right] \right\} \end{aligned} \tag{39}$$

and

$$C_{mn} = \sum_{m=1}^{\infty} \left\{ \frac{1}{2\alpha_m^4 \sinh(\alpha_m b)} - \frac{\sinh(\alpha_m b)}{\alpha_m^4 [\sinh(\alpha_m b) \cosh(\alpha_m b) + \alpha_m b]} \right. \\ \left. \times \left[\frac{e^{-\alpha_m^2 kt}}{\alpha_m b} + 2\alpha_m^3 b^3 \sum_{n=1}^{\infty} \frac{e^{-k \left[(n^2 \pi^2 + \alpha_m^2 b^2) \frac{t}{b^2} \right]}}{(n^2 \pi^2 + \alpha_m^2 b^2)^2} \right] \right\} \quad (40)$$

4 NUMERICAL CALCULATIONS

$$\text{Setting } f(r) = (r^2 - a^2)^2 \quad (41)$$

Applying finite Hankel transform as defined in equation (21) to the equation (41), one obtains

$$\bar{f}(\alpha_m) = \int_0^a r (r^2 - a^2)^2 J_0(\alpha_m r) dr, \\ \bar{f}(\alpha_m) = \frac{8a \{ (8 - a^2 \alpha_m^2) J_1(\alpha_m a) - 4a \alpha_m J_0(\alpha_m a) \}}{\alpha_m^5} \quad (42)$$

The numerical calculations has been carried out for different metal plates with parameters $a = 1$ m, $b = 0.2$ m, with

$$\alpha_1 = 3.8317, \alpha_2 = 7.0156, \alpha_3 = 10.1735, \alpha_4 = 13.3237, \alpha_5 = 16.470,$$

$$\alpha_6 = 19.6159, \alpha_7 = 22.7601, \alpha_8 = 25.9037, \alpha_9 = 29.0468, \alpha_{10} = 32.18$$

are the roots of transcendental equation $J_1(\alpha a) = 0$.

The numerical results have been compared for following metal plates,

Steel Plate (Carbon 1%) - thermal diffusivity $k = 11.72 \times 10^{-6} (m^2 s^{-1})$,

Iron Plate (Pure) - thermal diffusivity $k = 20.34 \times 10^{-6} (m^2 s^{-1})$,

Aluminum Plate (Pure) - thermal diffusivity $k = 84.18 \times 10^{-6} (m^2 s^{-1})$ and

Copper Plate (Pure) - thermal diffusivity $k = 112.34 \times 10^{-6} (m^2 s^{-1})$.

For convenience setting $X = \frac{16KQ}{10^5 a \lambda}$ and $Y = \frac{32GKQ}{10^5 a \lambda}$ in the expressions (33) to (38) and using equations (39), (40), and (41) one obtains the expressions for temperature, displacement and stress components.

In order to examine the influence of arbitrary heat flux on the upper and lower surface of thick plate, one performed numerical calculations in radial direction for $r = 0, 0.2, 0.4, 0.6, 0.8, 1$ keeping $z = 0.1$ at time $t = 5$. Also in axial direction for $z = -0.2, -0.1, 0, 0.1, 0.2$ keeping $r = 0.5$ at same time $t = 5$. Numerical variations in radial and axial directions for different metal plates are shown in the figures with the help of computer programme.

CONCLUDING REMARKS

In this paper a thick circular plates of different metals are considered which is subjected to a transient axisymmetric temperature field on the radial and axial directions of the cylindrical coordinate system and determined the expressions for temperature, displacements and stress functions, due to arbitrary heat flux on the upper and lower surface. As a special case mathematical model is constructed for $f(r) = (r^2 - a^2)^2$ and performed numerical calculations. The thermoelastic behavior is examined such as temperature change, displacements and stresses with the help of arbitrary heat flux on the upper and lower surface of plate. The numerical results are compared for different metal plates.

From figure 1 and 2, Radial displacement occurs throughout the plate in axial as well as in radial direction. It is zero at the center and at circular boundary surface ($r = 0, r = a$).

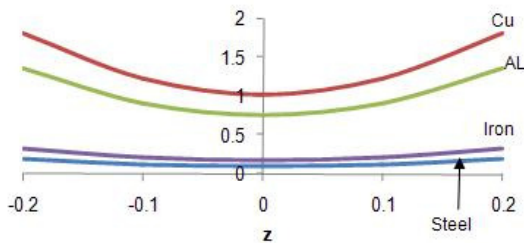


Figure 1: The radial displacement function u_r/X in axial direction at $r = 0.5$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

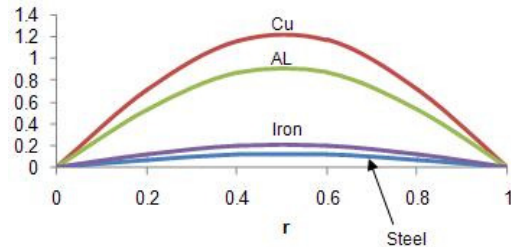


Figure 2: The radial displacement function u_r/X in radial direction at $z = 0.1$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

From figure 3 and 4, Axial displacement takes place at upper and lower surface the plate. It is zero at the middle within $-1 \leq z \leq 1$.

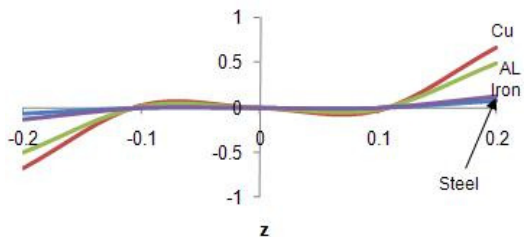


Figure 3: The axial displacement function u_z/X in axial direction at $r = 0.5$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

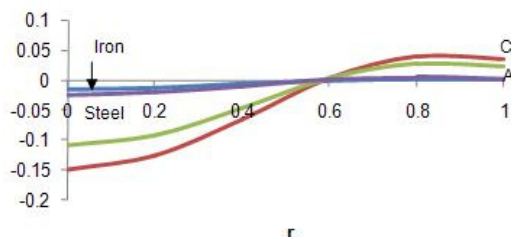


Figure 4: The axial displacement function u_z/X in radial direction at $z = 0.1$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

From figure 5 and 6, Radial stress function σ_r develops compressive stress in axial direction, where as in radial direction it develops tensile stress within annular region $0.8 \leq r \leq 1$ and compressive stress in circular region $0 \leq r \leq 0.8$.

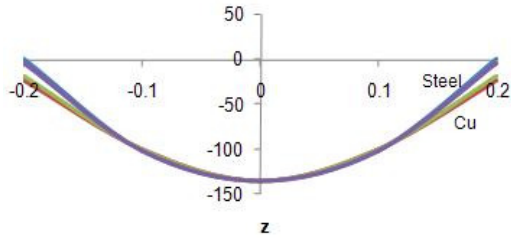


Figure 5: The radial stress function σ_r/Y axial direction at $r = 0.5$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

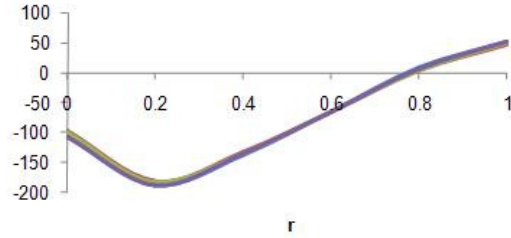


Figure 6: The radial stress function σ_r/Y radial direction at $z = 0.1$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

From figure 7 and 8, The stress function $\sigma_{\theta\theta}$ develops tensile stress within circular region $0 \leq r \leq 0.7$ and compressive stress in annular region $0.7 \leq r \leq 1$.

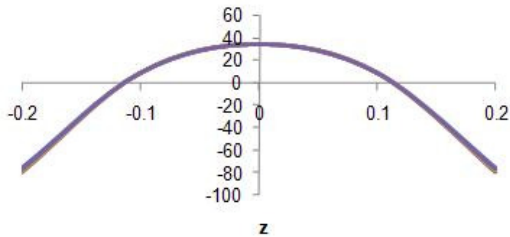


Figure 7: The stress function $\sigma_{\theta\theta}/Y$ axial direction at $r = 0.5$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

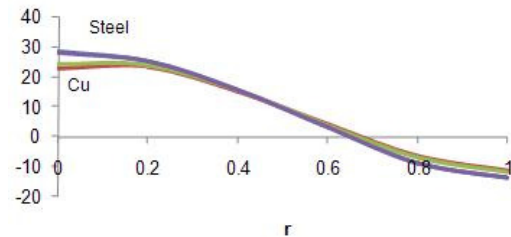


Figure 8: The stress function $\sigma_{\theta\theta}/Y$ radial direction at $z = 0.1$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

From figure 9 and 10, Axial stress function σ_{zz} develops tensile stress within circular region $0 \leq r \leq 0.7$ and compressive stress in annular region $0.7 \leq r \leq 1$, where as in axial direction it develops tensile stress.

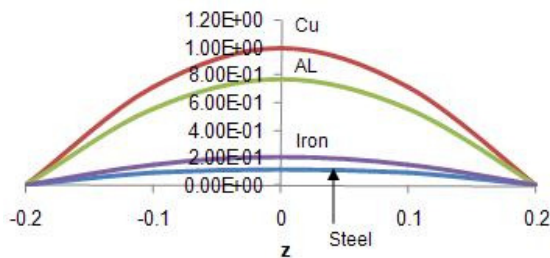


Figure 9: The axial stress function σ_{zz}/Y in axial direction at $r = 0.5$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

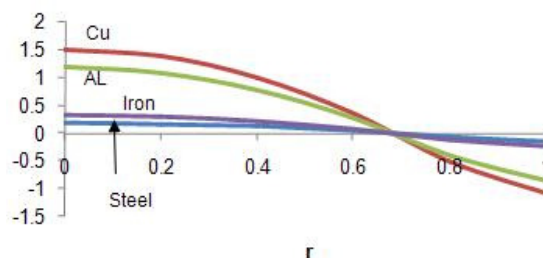


Figure 10: The axial stress function σ_{zz}/Y in radial direction at $z = 0.1$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

From figure 11, Stress function σ_{rz} develops tensile stress radial direction where as in axial direction it develops tensile stress at the upper half of plate and compressive stress at the lower half of plate.

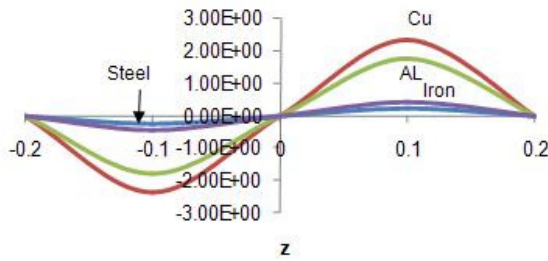


Figure 11: The stress function σ_{rz}/Y axial direction at $r = 0.5$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

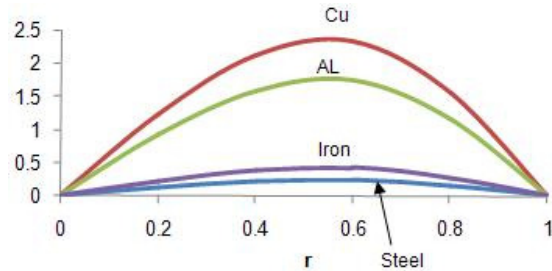


Figure 12: The stress function σ_{rz}/Y radial direction at $r = 0.5$. Cu-Copper (Pure), AL-Aluminum (Pure), Iron (Pure) and Steel (Carbon-1%).

It means we may find out that displacement and stress components occurs near heat source. The numerical values of the displacements and stresses for the plates of metals steel, iron, aluminum and copper are in the proportion and follows relation $Steel \leq Iron \leq Aluminum \leq Copper$. It means these values are directly proportional to their thermal diffusivity.

From the figures of radial and axial displacements it can observe that the radial displacement occur away from the center ($r = 0$). The axial displacement is in the downward direction at the center and upward direction near the circular boundary of the plate. Also from the figures of stress functions it can observe that the radial stress develops compressive stress and the axial stress develops tensile stress at the center of the plate, where as its opposite happens at the outer circular boundary of the plate. So it may conclude that due arbitrary heat flux on the upper and lower surfaces of plate expand in axial direction and bend concavely at the center. This expansion and bending is in the proportion of the thermal diffusivity of metal.

The results obtained here are more useful in engineering problems particularly in the determination of state of strain in thick circular plate. Also any particular case of special interest can be derived by assigning suitable values to the parameters and function in the expressions (33) to (38).

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REFERENCES

Hany H S and Farid AH (1994). Generalized thermoelastic problem of a thick plate under axisymmetric temperature distribution. *Journal of Thermal stresses*, 17(3), pp 435-452.

Ishihara M, Tanigawa Y, Ryuusuke K, and Noda N (1997). Theoretical analysis of thermoelastoplastic deformation of a circular plate due to a Partially distributed heat supply. *Journal of Thermal Stresses*, 20, pp 203-225.

Kulkarni VS and Deshmukh KC.(2007). Quasi-Static Thermal Stresses in a thick circular plate. *Applied Mathematical Modeling, Elsevier Publication*, 31(8), pp 1479-1488.

Noda N, Hetnarski RB. and Tanigawa Y (2003). *Thermal Stresses*. 2nd edn., pp 259-261, Taylor and Francis New York.

Rokne J, Singh BM, Dhaliwal RS, and Vrbik J (2004). The Axisymmetric Boussinesq type Problem for half space under optimal heating of arbitrary Profile. *IJMMS, Hindwai Publication*, 40, pp 2123-2131.

Sharma JN, Sharma PK, and Sharma RL(2004). Behavior of thermoelastic thick plate under lateral loads. *Journal of thermal stresses*, 27, pp 171-191, 2004.

Yoshihiro O (1996). Axisymmetric thermal stress analysis in the steady state with heat generation in the region by boundary element method. *Journal of thermal stresses*, 19 (3), pp 273-286.