ANALYTIC SOLUTION FOR CREEPING FLOW OF AN UNSTEADY MICROPOLAR FLUID

N. A. Khan\textsuperscript{1} and M. Jamil\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, University of Karachi, Karachi-75270, Pakistan. Email: njbalam@yahoo.com
\textsuperscript{2}Department of Mathematics & Basic Sciences, NEDUET, Karachi-75270, Pakistan.

Received 23 June 2007; accepted 23 September 2008

ABSTRACT

The two dimensional equations of motion for the slowly flowing of an unsteady incompressible micropolar fluid are written in Cartesian coordinates neglecting the inertial and gyroinertial terms. Using a transformation variable we are able to convert the governing equations into simple ordinary differential equations and then solved analytically. Finally graphs are plotted for physical interest.

Keywords: Micropolar fluid, exact solutions, microrotation, stream function.

1 INTRODUCTION

Eringen (Eringen 1964; Eringen 1966) developed the theory of micro fluid, which exhibits microscopic effects arising from the local structure and micro motions of the fluid elements. Such fluids support stress and body moments and include the local rotary inertia. The equations based on the theory of microfluids are much more complicated even for the case of a constitutively linear situation and the non-trivial solution in the field is not easy to obtain. There is a subclass of micro fluids namely the micropolar fluid for which one can reasonably hope to obtain non-trivial analytic solutions. Micropolar fluids are fluids with microstructure. They belong to a class of fluids with non-symmetric stress tensor and include, as a special case, the well-established Navier-Stokes model. Physically, micropolar fluids may represent fluid consisting of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, where the deformation of fluid particles is ignored. The model of micropolar fluids is worth studying as a very well balanced one. It is a well-founded significant generalization of the classical Navier-Stokes model.

Recently, the studies of micropolar fluid have acquired a special status due to their industrial application. Such applications include the extrusion of polymers fluid, colloidal fluids, real fluid with suspension, liquid crystals, cooling of metallic plate in a bath and animal blood.

The problem of finding exact solutions of governing equations of micropolar fluid flows present insurmountable difficulties due to the fact that these equations are non-linear. Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical, asymptotic...
methods. Although computer techniques make the complete integration of the equations of motion feasible, the accuracy of the results can be established by comparison with an exact solution.

Several investigators in the field have made the useful investigations that involve a micropolar fluid. For example Lyengar and Vani (Lyengar and Vani 2004) examined the flow of a micropolar fluid between two concentric spheres, induced by their rotary oscillations. El-Bary (El-Bary 2005) developed the exponential solution of the problem of two-dimensional motion of a micropolar fluid in a half plane. Dubey et al (Dubey et al. 1990) analyzed the flow of a micropolar fluid between two parallel plates rotating about two non-coincident axes under variable surface charges. Kim and Lee (Kim and Lee 2003) made an interesting study for the Hartman oscillatory flow problem of a micropolar fluid. Abo-Eldahab and Ghonaim (Abo-Eldahab and Ghonaim 2006) discussed the numerical solution in order to see the radiation effect on heat transfer of a micropolar fluid. Stavre (Stavre 2002) studied an optimal control problem associated with the motion of a micropolar fluid, with applications in the control of the blood pressure. Mostafa et al (Mostafa et al. 2006) investigate the effects of radiation on the boundary layer flow and the heat transfer of an electrically conducting micropolar fluid over a continuously moving stretching surface embedded in a non-Darcian porous medium a uniform magnetic field. Boundary layer on continuous surface is an important type of flow occurring in the number of technical problems. Examples may be found in continuous casting, glass fiber production, metal extrusion, hot rolling textiles and wire drawing. Adluri (Adluri 1995) presented hodographic study of a plane micropolar fluid. He transformed the equations to the hodograph plane by means of Legendre transformation function found exact solutions and geometry of the flow. Shahzad et al (Shahzad et al. 2007) presented analytic solution of an incompressible micropolar fluid using group method. However, most of the previous investigations deal with the numerical solution.

This paper concerns the unsteady flow of an incompressible micropolar fluid. The velocities are expressed in terms of a stream function and introducing a transformation variable derives expression for velocities and micro-rotation component.

The paper organized in the following fashion. In section 2, problem is formulated and governing equations and derived with their compatibility equations. In section 3, exact solution of an unsteady, incompressible micropolar fluid is discussed. Throughout section 2 and 3 we present the analysis in absence of body forces.

2 FORMULATION OF THE PROBLEM AND DISTURBANCE EQUATIONS

Assuming the flow to be Stokesian, neglecting the inertial and gyroinertial terms the field equations of the micropolar fluid dynamics are

\[
\text{div} \mathbf{V} = 0
\]

\[
\rho \frac{\partial \mathbf{V}}{\partial t} = - \text{grad} P + k \text{Curl} \mathbf{N} - (\mu + k) \text{Curl Curl} \mathbf{V}
\]
\[ \rho j \frac{\partial N}{\partial t} = 2kN + k \text{Curl} V - \gamma \text{Curl} \text{Curl} \overrightarrow{V} + (\alpha + \beta + \gamma) \text{grad} \left( \text{div} \overrightarrow{N} \right) \]  

(3)

where \( V \) is the velocity vector, \( N \) is the microrotation vector and the \( P \) is the fluid pressure, \( \rho \) and \( j \) are the fluid density and microgyration parameter, and \( \{ \mu, k \} \) and \( \{ \alpha, \beta, \gamma \} \) are viscosity and gyroviscosity coefficients.

The stress tensor \( t_{ij} \) and the couple stress tensor \( m_{ij} \) are given by

\[ t_{ij} = -P \delta_{ij} + (2\mu + k)e_{ij} + k\epsilon_{ijm}(\omega_{m} - V_{m}) \]  

(4)

\[ m_{ij} = \alpha N_{k,k} \delta_{ij} + \beta N_{i,j} + \gamma N_{j,j} \]  

(5)

where \( \omega \) is the vorticity vector and \( \delta_{ij} \) is the kronecker delta and \( \epsilon_{ijm} \) is the alternating symbol. We choose velocity vector \( \overrightarrow{V} \), microrotation \( \overrightarrow{N} \) and pressure \( P \) in the form

\[ \overrightarrow{V} = \left( u(x, y) e^{i\omega t}, v(x, y) e^{i\omega t}, 0 \right), \overrightarrow{N} = \left( 0, 0, C(x, y) e^{i\omega t} \right), P = p(x, y) e^{i\omega t} \]  

(6)

The equation of continuity in equation (1) implies the existence of the stream function \( \psi(x, y) \) such that

\[ u(x, y) = \psi_y, v(x, y) = -\psi_x \]  

(7)

Equations (2) and (3), on utilizing equations (6) and (7), we get

\[ i \rho \omega \psi_y = -p_x + k c_y + (\mu + k) \nabla^2 \psi_y \]  

(8)

\[ i \rho \omega \psi_x = p_y + k c_x + (\mu + k) \nabla^2 \psi_x \]  

(9)

\[ i \rho \omega j = -2k c - k \nabla^2 \psi + \gamma \nabla^2 c \]  

(10)

Eliminating pressure \( p \) from equations (8) and (9), we get

\[ \nabla^2 \left( \nabla^2 - \alpha^2 \right) \left( \nabla^2 - \beta^2 \right) \psi = 0 \]  

(11)

and

\[ k \left( i j \rho \omega + 2k \right) c = -\gamma (\mu + k) \nabla^4 \psi + \left( i \gamma \rho \omega - k^2 \right) \nabla^2 \psi \]  

(12)
where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplacian operator and

\[
\alpha^2 + \beta^2 = \frac{k(2 \mu + k) + i \omega \rho \left[ \gamma + j(\mu + k) \right]}{\gamma(\mu + k)} \\
\alpha^2 \beta^2 = \frac{i \rho \omega \left(2k + i \omega \rho \right)}{\gamma(\mu + k)}
\]

### 3 SOLUTION

Let

\[
\psi(x, y) = G(\xi)
\]

where

\[
\xi = ax + by
\]

On using the transformation equations (11) and (12), yields

\[
D^2 \left( D^2 - m^2 \right) \left( D^2 - n^2 \right) G(\xi) = 0
\]

\[
c = \lambda_1 D^4 \psi + \lambda_2 D^2 \psi
\]

where

\[
m^2 = \frac{\alpha^2}{a^2 + b^2}, n^2 = \frac{\beta^2}{a^2 + b^2}, \lambda_1 = \frac{-\gamma(\mu + k) \left( a^2 + b^2 \right)^2}{k \left( i \rho \omega + 2k \right)}
\]

\[
\lambda_2 = \frac{i \rho \omega - k^2}{k \left( i \rho \omega + 2k \right)} D = \frac{d}{d\xi}
\]

The solution of the equation (14) and using equation (13) gives

\[
\psi = A_1 + A_2 \xi + A_3 e^{m \xi} + A_4 e^{-m \xi} + A_5 e^{n \xi} + A_6 e^{-n \xi}
\]

Equation (15), on using equation (16) becomes

\[
c = c_1 \left( A_3 e^{m \xi} + A_4 e^{-m \xi} \right) + c_2 \left( A_5 e^{n \xi} + A_6 e^{-n \xi} \right)
\]

where
\[ c_1 = m^4 \lambda_1 + m^2 \lambda_2, c_2 = n^4 \lambda_1 + n^2 \lambda_2 \]

From equations (8) to (9), (16) to (17), we have

\[ p_x = -i \omega b \rho A_2 - m b^2 a \lambda_3 \left( A_3 e^{m \xi} - A_3 e^{-m \xi} \right) + n b^2 a \lambda_4 \left( A_3 e^{n \xi} - A_3 e^{-n \xi} \right) \tag{18} \]

\[ p_y = i \omega \rho a A_2 + m b \lambda_3 \left( A_3 e^{m \xi} - A_3 e^{-m \xi} \right) + n b \lambda_4 \left( A_3 e^{n \xi} - A_3 e^{-n \xi} \right) \tag{19} \]

On solving the equations (18) and (19), we get

\[ p = \lambda_3 \left( A_1 e^{n(ax+by)} + A_4 e^{-m(ax+by)} \right) + \lambda_4 \left( A_5 e^{n(ax+by)} + A_4 e^{-n(ax+by)} \right) \]

\[ + i \rho \omega A_2 \left( a y - b x \right) + A_7 \tag{20} \]

where

\[ \lambda_3 = \frac{a}{b} \left( i \omega \rho - k c_1 - m^2 \left( \mu + k \right) \right) \left( a^2 + b^2 \right) \]

\[ \lambda_4 = \frac{a}{b} \left( i \omega \rho - k c_2 - m^2 \left( \mu + k \right) \right) \left( a^2 + b^2 \right) \]

The expression for \( u, v \) and \( c \) are given by

\[ u(x, y) = A_2 b + m b \left( A_1 e^{m(ax+by)} - A_4 e^{-m(ax+by)} \right) + n b \left( A_5 e^{n(ax+by)} - A_6 e^{-n(ax+by)} \right) \tag{21} \]

\[ v(x, y) = - A_2 b - m a \left( A_3 e^{n(ax+by)} - A_4 e^{-m(ax+by)} \right) - n a \left( A_5 e^{n(ax+by)} + A_6 e^{-n(ax+by)} \right) \tag{22} \]

\[ c = c_1 \left( A_1 e^{m(ax+by)} + A_4 e^{-m(ax+by)} \right) + c_2 \left( A_5 e^{n(ax+by)} + A_6 e^{-n(ax+by)} \right) \tag{23} \]

4 RESULTS AND DISCUSSION

Here, the solution for the time-independent \( x \) and \( y \)-components of velocity distribution and microrotation are plotted along the \( y \)-direction for various values of \( m \) and \( n \). Because \( m \) and \( n \) are directly proportional to gyroviscosity coefficients \( \alpha \) and \( \beta \) therefore these graphs are shown the variation of components of velocity distribution and microrotation with respect to gyroviscosity coefficients \( \alpha \) and \( \beta \) along \( y \)-axis. Figures 1, 2, 4, and 5 have been prepared for the velocity components whereas figures 3 and 6 hold for microrotation.
It is found from figures 1 and 3 that the velocity component $u$ and microrotation $c$ are increasing function of $m$. It is also evident from figure 2 that the behavior of $m$ on the velocity component $v$ is opposite to that of $u$ and $c$. Similarly from figures 4 and 6, it is found that $x$-component of velocity distribution and microrotations are increase by increasing the value of $m$. It is clear from figure 5 that the behavior of $y$-component of velocity is opposite to that of the $x$-component of velocity distribution and microrotation.

Figure 1: Variation of $x$-component of velocity distribution along $y$-axis with the value of $m$ ($A_2=5$, $A_3=3$, $A_4=4$, $A_5=2$, $A_6=4$, $a=3$, $b=2$, $n=3$)

Figure 2: Variation of $y$-component of velocity distribution along $y$-axis with the value of $m$ ($A_2=5$, $A_3=3$, $A_4=4$, $A_5=2$, $A_6=4$, $a=3$, $b=2$, $n=3$)

Figure 3: Variation of microrotation along $y$-axis with the value of $m$ ($A_2=5$, $A_3=3$, $A_4=4$, $A_5=2$, $A_6=4$, $a=3$, $b=2$, $n=3$)

Figure 4: Variation of $x$-component of velocity distribution along $y$-axis with the value of $n$ ($A_2=5$, $A_3=3$, $A_4=4$, $A_5=2$, $A_6=4$, $a=3$, $b=2$, $m=3$)
Finally from figures 1 to 6 we conclude that x-component of velocity distribution $u$ and microrotation $c$ are increasing function of gyroviscosity coefficients $\alpha$ and $\beta$, whereas $y$-component of velocity distribution $v$ is opposite to that of $x$-component of velocity distribution $u$ and microrotation $c$ along $y$-axis. Similarly same conclusion can be drawn along $x$-axis and $\xi$-axis.

5 CONCLUDING REMARKS

The unsteady two-dimensional equations of motion for creeping flow of an incompressible micropolar fluid are considered neglecting the inertial and gyroinertial terms. Employing transformation variable the governing equations reduces to ordinary differential equations and then solved analytically. Several graph of physical interest are also displayed and discussed. Graphs are shown that the variation of gyroviscosity coefficients has a significant influence on the time independent of velocity distribution and microrotation. The findings are summarized as follows:

1. The $x$-component of velocity distribution $u$ and microrotation $c$ is increasing function of gyroviscosity coefficients along $y$-axis.
2. The $y$-component of velocity distribution $v$ is decreasing function of gyroviscosity coefficients along $y$-axis.
3. Similar conclusion can be drawn along $x$-axis and $\xi$-axis.

NOMENCLATURE

$V$ Unsteady Velocity
$u, v$ Steady velocity components
$P$ Unsteady pressure
$P$ Steady pressure
$\mu, k$ Viscosity

\[ \rho \] Density of fluid
\[ \alpha, \beta, \gamma \] Gyroviscosity coefficients
\[ \psi \] Stream function
\[ G \] Function
\[ \omega \] Frequency
\[ \lambda_1, \ldots, \lambda_4 \] Constants
\[ m, n \] Real constants
\[ A_1, \ldots, A_6 \] Real constants

REFERENCES


