CHEBYSHEV POLYNOMIALS AND FREDHOLM-VOLTERRA INTEGRAL EQUATION

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ABSTRACT

In this article, we discuss the existence and uniqueness solution of FVIE of the second kind. Also, using Chebyshev polynomial, the solution of the integral equation is obtained. The kernel of Fredholm, in position, has a smooth term multiplying by a logarithmic function, while the kernel of Volterra is continuous function in time. Some numerical results are obtained using maple 7.

Keywords: Fredholm-Volterra integral equation (FVIE), Chebyshev polynomial, logarithmic kernel, linear algebraic system.

1 INTRODUCTION

Many problems of mathematical physics, theory of elasticity and contact problems reduce to an integral equations with singular kernel. Singular integral equation appears in studies involving fracture mechanics (Aleksandrov and Kovalenko 1986; Willis and Nasser 1990), radiation and molecular conduction (Frankel 1995) and others. Over the past thirty years, the substantial progress has been made in developing innovative approximate analytical and purely numerical solution to a large class of integral equations with singular kernel. Since closed form solutions of these problems are generally not available, much attention has been focused on numerical treatment. In (Abdou 2000) Abdou, using orthogonal polynomial method of type Jacobi polynomial, obtained the solution of FIE of the second kind with Cauchy and logarithmic kernel. Also, Abdou and Hassan in (Abdou and Hassan 2000), using Legendre polynomial, obtained the solution of FIE with logarithmic kernel. Using Toeplitz matrices method, Abdou et. al. (Abdou et al. 2002), obtained the solution of FIE with logarithmic and Carleman kernels. While in (Abdou 2002a), Abdou using orthogonal polynomial method of type Legendre polynomial, the solution of FVIE is obtained.

This paper is divided into seven sections, the formulation of FVIE of the second kind which investigated from the contact problem is considered in section two. While, in section three, the
existence and uniqueness solution of the integral equation is discussed and proved. In section four, we state some algebraic and integral formulas for the Chebyshev polynomials. In section five and six, we use Chebyshev polynomial method to obtain a linear system of Volterra integral equation of the second kind. In the light of a numerical method, the system of Volterra equation reduced to a linear algebraic system. In section seven, numerical results are obtained and discussed.

2 FORMULATION OF THE PROBLEM

Consider the FVIE of second kind

\[ \mu \phi(x, t) + \lambda \int_0^1 V(|t - \tau|)\phi(x, \tau)d\tau + \lambda \int_{-1}^{1} p(x, y)k\left(\frac{x-y}{c}\right)\phi(y, t)dy = f(x, t), \]

\[ k(t) = \frac{1}{2} \int_{-\infty}^{\infty} \tanh u e^{iut} du, \quad t = \frac{x-y}{c}, \]

under the condition

\[ \int_{-1}^{1} \phi(t, y)dy = p(t), \quad t \in [0, T], \]

where \( \lambda \) is a constant, may be complex, and has many physical meaning. The constant \( \mu \) defined the kind of integral equation where, for \( \mu = 0 \), we have a Fredholm-Volterra integral equation of the first kind, \( \mu = \text{constant} \neq 0 \), for the second kind, and \( \mu = \mu(x) \) for the third kind. The given function of time \( V(|t - \tau|) \) represents the kernel of VI term and belongs to the class \( C[0, T] \), where \( t, \tau \in [0, T], T < \infty \). The kernel of position, \( p(x, y)k\left(\frac{x-y}{c}\right) \), of FI term behaved badly for \( k\left(\frac{x-y}{c}\right) \), given by Eq.(3), and smooth for \( p(x, y) \). The given function \( f(x, t) \) belongs to the space \( L_2[-1, 1] \times C[0, T] \).

The bad term of the kernel can be written in the form, see (Abdou and Hassan 2000)

\[ k(t) = -\ln \tanh \frac{\pi t}{4}, \quad t = \frac{x-y}{c}. \]

If \( c \to \infty \), and \( t \) is very small, so it satisfies that \( \tanh t \simeq t \), then we may write

\[ \ln |\tanh \frac{\pi t}{4}| \simeq \ln |x-y| - d, \quad d = \ln \frac{4c}{\pi}. \]

Hence, we have

\[ k\left(\frac{x-y}{c}\right) = -(\ln |x-y| - d). \]

Special cases:
(i) Let, in Eq.(1), \( t = 0 \), we have

\[ \mu \psi(x) + \lambda \int_{-1}^{1} p(x, y)k\left(\frac{x-y}{c}\right)\psi(y) = g(x), \quad (\psi(x) = \phi(x, 0), \; g(x) = f(x, 0)). \]

The integral equation (4) is called Fredholm integral equation of the second kind, the asymptotic solution of Eq.(4) when the kernel takes a logarithmic form is obtained by Abdou, see (Abdou 2002b).

(ii) In Eq.(4), let \( p(x, y) = 1 \) and \( k(|x - y|) = \ln|x - y| - d \), then differentiating the result, we have

\[
\mu \frac{d\psi(x)}{dx} - \lambda \int_{-1}^{1} \frac{\psi(y)}{x - y} dy = h(x), \quad h(x) = \frac{dg(x)}{dx}.
\] (5)

The formula (5) represents an integro-differential equation with Cauchy kernel. Taking the transformations \( x = 2z - 1 \) and \( y = 2\eta - 1 \), in (5), we get

\[
\mu \frac{d\Psi(z)}{dz} - \lambda \int_{0}^{1} \frac{\Psi(\eta)}{z - \eta} d\eta = h(z).
\] (6)

The formula (6) has appeared in both combined infrared gaseous radiation and molecular conduction and elastic constant studies.

When \( \mu = 1 \) and \( h(z) = z \), Eq.(6) is solved and discussed numerically by Frankel (Frankel 1995). Also, when \( \mu = 0 \), we have an integral equation of the first kind with Cauchy kernel which appears in airfoil theory and combined infrared radiation in molecular conditions and in contact problems, see (Venturino 1992; Muskelishvili 1953).

(iii) When \( p(x, y) = 1 \) and the kernel of position takes a logarithmic form, the solution of integral equation (1) is obtained numerically using Legendre polynomial, see Abdou and Nasr (Abdou and Nasr 2003).

3 EXISTENCE AND UNIQUENESS SOLUTION

In order to discuss the existence and uniqueness solution of Eq.(1), with the aid of Eq.(2), we assume the following:

(i) The bad behaved kernel satisfies the discontinuity condition

\[
\int_{-1}^{1} \int_{-1}^{1} |k(x - y)|^2 dx dy = C^2 < \infty,
\]

while the smooth kernel \( |p(x, y)| < N_1 \).

(ii) The positive kernel of time is continuous and satisfies

\( V(|t - \tau|) < N_2, \quad \forall t, \tau \in [0, T] \).

(iii) The given continuous function \( f(x, t) \in L_2[-1, 1] \times C[0, T] \) and its norm is given as

\[
\| f \|_{L_2 \times C} = \max_{0 \leq t \leq T} \int_{-1}^{1} \left[ \int_{-1}^{1} f^2(x, \tau) dx \right]^\frac{1}{2} d\tau \leq M
\]

(iv) The unknown function \( \phi(x, t) \) satisfies Lipschitz condition with respect to position

\[
|\phi(x_1, t) - \phi(x_2, t)| \leq A_1(t)|x_1 - x_2|,
\]

and Holder condition with respect to time

\[
|\phi(x, t_1) - \phi(x, t_2)| \leq A_2(x)|t_1 - t_2|^{\alpha}, \quad 0 \leq \alpha < 1.
\]

**Theorem 3.1:**

The solution of the integral equation (1) is exist and unique, under the condition

\[
|\lambda| \leq \frac{|\mu|}{N_2T + N_1C}.
\] (7)
Proof:
The existence of the solution will be obtained using successive approximation that will pick up any real continuous function \( \phi_0(x, t) \) in \( L_2[-1, 1] \times C[0, T] \), then let \( \mu \phi_0(x, t) = f(x, t) \) and construct a sequence \( \phi_n \) to obtain

\[
\mu \phi_n(x, t) = f(x, t) - \lambda \int_0^t \int_{-1}^1 p(x, y) k\left(\frac{x - y}{c}\right) \phi_{n-1}(y, t) dy d\tau - \lambda \int_{-1}^1 \phi_{n-1}(x, \tau) d\tau \tag{8}
\]

It is convenient to introduce

\[
\psi_n(x, t) = \phi_n(x, t) - \phi_{n-1}(x, t), \quad (\mu \psi_0(x, t) = f(x, t)),
\]

then, we write

\[
\phi_n(x, t) = \sum_{i=0}^n \psi_i(x, t). \tag{10}
\]

Hence, we get

\[
\mu \psi_n(x, t) = -\lambda \int_0^t \int_{-1}^1 p(x, y) k\left(\frac{x - y}{c}\right) \psi_{n-1}(y, t) dy d\tau + \int_{-1}^1 p(x, y) k\left(\frac{x - y}{c}\right) \psi_{n-1}(y, t) dy \] \tag{11}

Using Cauchy-Schwarz inequality, one obtains

\[
\| \psi_n(x, t) \| \leq \left| \frac{\lambda}{\mu} \right| [N_2T + N_1C] \| \psi_{n-1}(x, t) \|, \quad T = \text{max } t. \tag{12}
\]

By induction, we have

\[
\| \psi_n(x, t) \| \leq \left| \frac{\lambda}{\mu} \right|^n [N_2T + N_1C]^n M. \tag{13}
\]

Therefore, \( \psi_n(x, t) \) are convergent, then the formula (10) is also convergent. This leads us to write

\[
\phi(x, t) = \sum_{i=0}^\infty \psi_i(x, t). \tag{14}
\]

The formula (14) leads us to say that \( \phi(x, t) \) exists and represents a continuous solution.

To prove the uniqueness, assume \( \tilde{\phi}(x, t) \) is another solution. Hence, we get

\[
\| \tilde{\phi}(x, t) - \phi(x, t) \| = \left| \frac{\lambda}{\mu} \right| \int_0^t \int_{-1}^1 p(x, y) k\left(\frac{x - y}{c}\right) \tilde{\phi}(y, t) - \phi(y, t) dy d\tau + \left| \frac{\lambda}{\mu} \right| \int_{-1}^1 p(x, y) k\left(\frac{x - y}{c}\right) \tilde{\phi}(y, t) - \phi(y, t) dy \right|, \tag{15}
\]

therefore

\[
\| \tilde{\phi}(x, t) - \phi(x, t) \| \leq \left| \frac{\lambda}{\mu} \right| \int_0^t \int_{-1}^1 p(x, y) k\left(\frac{x - y}{c}\right) \tilde{\phi}(y, t) - \phi(y, t) dy d\tau \] \tag{16}

Using Cauchy-Schwarz inequality, we get

\[
\| \tilde{\phi}(x, t) - \phi(x, t) \| \leq \left| \frac{\lambda}{\mu} \right| (N_2T + N_1C) \| \tilde{\phi}(x, t) - \phi(x, t) \| \], \tag{17}

Using (7), we see that \( \phi = \tilde{\phi} \).
4 INTEGRAL OPERATOR

The integral equation (1) can be written in the integral operator form as

\[ K \phi = \phi, \quad K \phi = \frac{1}{\mu} (f(x,t) - \lambda L \phi), \quad (\mu \neq 0) \tag{18} \]

where

\[ L \phi = \int_0^t V(|t - \tau|) \phi(x,\tau) d\tau + \int_{-1}^1 p(x,y) k\left(\frac{x-y}{c}\right) \phi(y,t) dy. \tag{19} \]

Here, we use a Banach fixed point theorem. For this, the normality and continuity of the integral operator (18) must be discussed and proved.

Normality:
The normality of the integral operator \( L \), defined by (19), can be shown as

\[ \| L \phi \| \leq |\lambda| \| L \phi \| \leq |\lambda| \left( N_2 T + N_1 C \right) \| \phi \|. \tag{20} \]

Using Cauchy-Schwarz inequality, we get

\[ \| L \phi \| \leq (N_2 T + N_1 C) \| \phi \|. \tag{21} \]

So, the norm of \( L \) is given by

\[ \| L \| = \sup_{\phi \neq 0} \frac{\| L \phi \|}{\| \phi \|}. \tag{22} \]

The formula (22) proves the normality of the FVIE of the first kind, \( i.e. \) when \( \mu = 0 \).
When \( \mu \neq 0 \), the inequality (22) leads us to say that the integral operator of (18) has a normality, where

\[ \| K \phi \| \leq \frac{1}{|\mu|} (H + |\lambda|(N_2 T + N_1 C) \| \phi \|). \tag{23} \]

Continuity:
To prove the continuity of the integral operator (18), we assume that the two functions \( \phi_n(x,t) \), \( \phi_m(x,t) \) satisfies (18), then we have

\[ \| K \phi_n - K \phi_m \| = \frac{\lambda}{\mu} \| L \phi_n - L \phi_m \| = \frac{\lambda}{\mu} \left| \int_0^t V(|t - \tau|) (\phi_n(x,\tau) - \phi_m(x,\tau)) d\tau \right| \]

\[ + \int_{-1}^1 p(x,y) k\left(\frac{x-y}{c}\right) (\phi_n(x,\tau) - \phi_m(x,\tau)) dy \]. \tag{24} \]

Therefore, we get

\[ \| K \phi_n - K \phi_m \| \leq \alpha \| \phi_n - \phi_m \| \quad \alpha = \frac{\lambda}{\mu} (N_2 T + N_1 C). \tag{25} \]

If \( \| \phi_n - \phi_m \| \to 0 \) then \( \| K \phi_n - K \phi_m \| \to 0 \), which prove the continuity, hence the integral operator \( K \) is a contraction mapping then, by fixed point theorem, Eq.(1) has a unique solution in \( L_2[-1,1] \times C[0,T] \).

5 CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials of degree $n$ of the first kind are defined by

$$T_n(x) = \cos(n \cos^{-1} x), \quad |T_n(x)| < 1, \quad n = 0, 1, 2, \ldots$$

For convenience, we state some well known algebraic and integral relations associated with Chebyshev polynomials, see (Erezyli et al. 1985; Gradshteyn and Ryzhik 1994)

1) Algebraic formula:

$$T_m(x)T_n(x) = \frac{1}{2} [T_{m+n}(x) + T_{|m-n|}(x)].$$

2) Integral relations:

(i) \[ \int_{-1}^{1} T_n(x)^2 \, dx = \begin{cases} \frac{\pi}{2}, & n = 0, 2, 4, \ldots \\ \frac{\pi}{n}, & n = 1, 3, 5, \ldots \end{cases} \]

(ii) \[ \int_{-1}^{1} T_n(x) \sqrt{1 - x^2} \, dx = \begin{cases} \frac{\pi}{2}, & n = 0 \\ \pi, & n \geq 1 \end{cases} \]

3) Orthogonality relation:

$$\int_{-1}^{1} T_n(y)T_m(y) \sqrt{1 - y^2} \, dy = \begin{cases} 0, & n \neq m \\ \frac{\pi}{2}, & n = m \neq 0 \\ \pi, & n = m = 0 \end{cases}$$

This orthogonal relation enables us to expand a function into Chebyshev series of the form

$$F(x, t) = \sum_{n=0}^{\infty} f_n(t)T_n(x),$$

where the coefficients $f_n(t)$ of Chebyshev polynomial belongs to $C[0, t]$, and can be obtained as the following

$$f_n(t) = \begin{cases} \frac{2}{\pi} \int_{-1}^{1} \frac{F(x, t)T_n(x)}{\sqrt{1-x^2}} \, dx, & n \neq 0 \\ \frac{1}{\pi} \int_{-1}^{1} \frac{F(x, t)}{\sqrt{1-x^2}} \, dx, & n = 0 \end{cases}$$

6 METHOD OF SOLUTION

In this section, we use Chebyshev polynomial $T_n(x)$ of the first kind of order $n$ to solve the integral equation (1). For this, we use (3) in Eq.(1) to get

$$\mu\phi(x, t) + \lambda \int_{0}^{1} V(|t - \tau|)\phi(x, \tau) \, d\tau - \lambda \int_{-1}^{1} p(x, y)(\ln |x-y| - d)\phi(y, t) \, dy = f(x, t),$$

The weight function which characterize the singular behavior of $\phi(x, t)$ in Eq.(1) is given by

$$R(x) = (1 + x)^{-\frac{1}{2} + \alpha}(1 - x)^{\frac{1}{2} + \beta},$$

where $\alpha, \beta = 0, 1, -1; \quad -1 < -\frac{1}{2} + \alpha < 1; \quad -1 < \frac{1}{2} + \beta < 1.$

Assume the unknown function $\phi(x, t)$, in the light of weight function, takes the form

$$\phi(x, t) = R(x)G(x, t), \quad R(x) = (1 - x^2)^{-\frac{1}{2}}.$$
where $G(x, t)$ is unknown function and $R(x)$ represents the weight function of $T_n(x)$. Now, to obtain numerically the solution of Eq.(33), under the condition (2), we express $G(x, t)$ as

$$G(x, t) = \sum_{n=0}^{\infty} a_n(t)T_n(x). \quad (36)$$

Hence, we have

$$\phi(x, t) = \sum_{n=0}^{\infty} a_n(t) \frac{T_n(x)}{\sqrt{1-x^2}}. \quad (37)$$

The formula (37) can be truncated to

$$\phi_N(x, t) = \sum_{n=0}^{N} a_n(t) \frac{T_n(x)}{\sqrt{1-x^2}}. \quad (38)$$

Since, any smooth function can be represented in the polynomial series form, therefore we assume the smooth function $p(x, y)$ of Eq.(33) in the form

$$p(x, y) = \sum_{m=0}^{M} T_m(x)T_m(y), \quad m = 0, 1, 2, 3, \ldots, M. \quad (39)$$

Using (39) and (38), Eq.(33) becomes

$$\mu \sum_{n=0}^{N} a_n(t) \frac{T_n(x)}{\sqrt{1-x^2}} - \lambda \sum_{n=0}^{N} \int_{0}^{t} a_n(\tau)V(|t-\tau|) \frac{T_n(x)}{\sqrt{1-x^2}} d\tau$$

$$- \lambda \sum_{n=0}^{N} \sum_{m=0}^{M} T_m(x) \int_{-1}^{1} a_n(t)T_m(y)(\ln |x-y| - d) \frac{T_n(y)}{\sqrt{1-y^2}} dy = \sum_{n=0}^{N} f_n(t) \frac{T_n(x)}{\sqrt{1-x^2}}, \quad (40)$$

where

$$f_n(t) = \begin{cases} \frac{2}{\pi} \int_{-1}^{1} f(x, t)T_n(x) dx, & n \neq 0 \\ \frac{1}{\pi} \int_{-1}^{1} f(x, t) dx, & n = 0 \end{cases} \quad (41)$$

After solving Eq. (40), the approximate solution is obtained directly from Eq.(38) then the exact solution is obtained from Eq.(37).

This method is said to be convergent of order $r$ if and only if for $N$ sufficiently large, there exists a constant $D > 0$ independent of $N$ such that

$$\| \phi(x, t) - \phi_N(x, t) \| \leq DN^{-r}. \quad (42)$$

So, the transformation error $E_1$ can be determined as

$$E_1 = \| \phi(x, t) - \phi_N(x, t) \| \leq \| \sum_{n=N+1}^{\infty} a_n(t) \|. \quad (43)$$

In the aid of (42), we write

$$E_1 \leq D(T)N^{-r}, \quad T = \max_i. \quad (44)$$

The solution of Eq.(40) can be obtained after discussing the following:

1. For \( n = 0, \ m = 0 \), and using (29), Eq.(40) yields

\[
\frac{\mu a_0(t)}{\sqrt{1 - x^2}} - \lambda \int_0^t a_0(\tau)V(|t - \tau|)d\tau - \lambda \pi (\ln 2 - d)a_0(t) = \frac{f_0(t)}{\sqrt{1 - x^2}}
\] (45)

Integrating over \( x \) from -1 to 1, we get

\[
[\mu - 2\lambda (\ln 2 - d)]a_0(t) - \lambda \int_0^t a_0(\tau)V(|t - \tau|)d\tau = f_0(t).
\] (46)

The formula (46) represents a Volterra integral equation of the second kind with continuous kernel.

2. For \( n = 0, \ m \neq 0 \), after using (29), Eq.(40) tends to

\[
\frac{\mu a_0(t)}{\sqrt{1 - x^2}} - \lambda \int_0^t a_0(\tau)V(|t - \tau|)d\tau - \pi \lambda \sum_{m=1}^{M} \frac{T_m(x)T_m(x)}{m}a_0(t) = \frac{f_0(t)}{\sqrt{1 - x^2}}
\] (47)

Integrating over \( x \) from -1 to 1, then using (27) and (28), one can obtain

\[
[\mu - 2\lambda \sum_{m=1}^{M} \frac{1 - 2m^2}{m(1 - 4m^2)}]a_0(t) - \lambda \int_0^t V(|t - \tau|)a_0(\tau)d\tau = f_0(t).
\] (48)

3. For \( n \neq 0, \ m = 0 \), we have

\[
[\mu - 2\lambda \sum_{l=1}^{N} A_{n,l}]a_n(t) - \lambda \int_0^t V(|t - \tau|)a_n(\tau)d\tau = f_n(t), \quad (n \geq 1)
\] (49)

where

\[
A_{n,l} = \begin{cases} 
\frac{1}{1-(n+l)^2} + \frac{1}{1-n^2}, & n + l = \text{even} \\
0, & n + l = \text{odd}
\end{cases}
\] (50)

4. For \( n = m \neq 0 \), we get

\[
[\mu - \lambda \sum_{l=1}^{N} \frac{A_{n,l}}{4n} + \beta_{n,l}(\ln 2 - d)]a_n(t) - \lambda \int_0^t V(|t - \tau|)a_n(\tau)d\tau = f_n(t), \quad (n, m \geq 1)
\] (51)

where

\[
A_{n,l} = \begin{cases} 
\frac{1}{1-(3n+l)^2} + \frac{1}{1-(3n-l)^2} + \frac{1}{1-n^2} + \frac{1}{1-n^2}, & n + l = \text{even} \\
0, & n + l = \text{odd}
\end{cases}
\] (52)

and

\[
\beta_{n,l} = \begin{cases} 
\frac{1}{1-(n+l)^2} + \frac{1}{1-(n-l)^2}, & n + l = \text{even} \\
0, & n + l = \text{odd}
\end{cases}
\] (53)

5. For \( n \neq m \neq 0 \), one has

\[
[\mu - \lambda \sum_{l=1}^{N} \sum_{m=1}^{M} A_{n,m,l}]a_n(t) - \lambda \int_0^t V(|t - \tau|)a_n(\tau)d\tau = f_n(t), \quad (n \geq 1, m \geq 1, n \neq m)
\] (54)
where

\[
A_{n,m,l} = \begin{cases} 
\frac{1}{m+n} \left( \frac{1}{1-(2m+n+l)^2} + \frac{1}{1-(2m+n-l)^2} \right) + \frac{1}{|m-n|} \left( \frac{1}{1-(2m-n+l)^2} + \frac{1}{1-(2m-n-l)^2} \right), & n + l = \text{even} \\
0, & n + l = \text{odd}.
\end{cases}
\]  

(55)

The integral equation (54) represents a system of Volterra integral equations of the second kind with continuous kernel.

**Convergence of the Algebraic system**

The convergence of the algebraic system can be derived from the following:

\[
\sum_{n,l=1}^{\infty} \sum_{m=1}^{M} A_{n,m,l} = \sum_{n,l=1}^{\infty} \sum_{m=1}^{M} \frac{1}{m+n} (L_{n,m,l}^{(1)} + L_{n,m,l}^{(2)} + L_{n,m,l}^{(3)}) + \sum_{n,l=1}^{\infty} \sum_{m=1}^{M} \frac{1}{|m-n|} (L_{n,m,l}^{(4)} + L_{n,m,l}^{(5)} + L_{n,m,l}^{(6)})
\]

(56)

Applying Cauchy-Minkowski inequality, we have

\[
\| A_{n,m,l} \| = \left( \sum_{n=1}^{\infty} \sum_{m=1}^{M} A_{n,m,l}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{n=1}^{\infty} \sum_{m=1}^{M} (\frac{1}{m+n})^2 \right)^{\frac{1}{2}} \left( \sum_{l,n=1}^{\infty} \sum_{m=1}^{M} (L_{n,m,l}^{(1)})^2 \right)^{\frac{1}{2}}
\]

\[
+ \left( \sum_{l,n=1}^{\infty} \sum_{m=1}^{M} (L_{n,m,l}^{(2)})^2 \right)^{\frac{1}{2}} + \left( \sum_{l,n=1}^{\infty} \sum_{m=1}^{M} (L_{n,m,l}^{(3)})^2 \right)^{\frac{1}{2}} + \left( \sum_{l,n=1}^{\infty} \sum_{m=1}^{M} (L_{n,m,l}^{(4)})^2 \right)^{\frac{1}{2}}
\]

\[
+ \left( \sum_{l,n=1}^{\infty} \sum_{m=1}^{M} (L_{n,m,l}^{(5)})^2 \right)^{\frac{1}{2}} + \left( \sum_{l,n=1}^{\infty} \sum_{m=1}^{M} (L_{n,m,l}^{(6)})^2 \right)^{\frac{1}{2}} \]  

(57)

where

\[
L_{n,m,l}^{(1)} = \frac{1}{1-(2m+n+l)^2}, \quad L_{n,m,l}^{(2)} = \frac{1}{1-(2m+n-l)^2}, \quad L_{n,m,l}^{(3)} = \frac{1}{1-(n+l)^2}
\]

(58)

and

\[
L_{n,l}^{(4)} = \frac{1}{1-(n-l)^2}, \quad L_{n,m,l}^{(5)} = \frac{1}{1-(2m-n+l)^2}, \quad L_{n,m,l}^{(6)} = \frac{1}{1-(2m-n-l)^2}
\]

(59)

Since \((\sum_{n=1}^{\infty} \sum_{m=1}^{M} (\frac{1}{m+n})^2)^{\frac{1}{2}}\) and \((\sum_{n=1}^{\infty} \sum_{m=1}^{M} (\frac{1}{m-n})^2)^{\frac{1}{2}} \) behaves like \((\sum_{n=1}^{\infty} \sum_{m=1}^{M} (\frac{1}{n})^2)^{\frac{1}{2}}\),

\[
L_{n,m,l}^{(1)} = \frac{1}{1-(2m+n+l)^2} < \frac{1}{(2m+n+l)^2} < \frac{1}{n^2}, \quad \text{and} \quad (\sum_{n=1}^{\infty} \sum_{m=1}^{M} (\frac{1}{n})^2)^{\frac{1}{2}} \text{ is convergent, therefore}
\]

\[
A_{n,m,l} \rightarrow 0 \quad \text{when} \quad n = l = \infty.
\]

**7 VOLTERRA INTEGRAL EQUATION**

Many different methods can be used to solve the Volterra integral equation of the second kind with continuous kernel, see Linz (Linz 1985), and Tricomi (Tricomi 1985). Here we use a numerical method to solve the Volterra integral equation of the second kind, Eq.(46). For this, we divide the interval time \([0, T]\) by

\[
0 \leq t_0 < t_1 < t_2 < \ldots < t_{\gamma} < \ldots < t_{\phi} = T, \quad 0 \leq \gamma \leq \phi
\]

(60)

Hence, using a suitable quadrature rule, and the following notations,

\[
a_n(t_\gamma) = a_n^{(\gamma)}, \quad f_n(t_\gamma) = f_n^{(\gamma)}, \quad V(|t_{\gamma} - t_k|) = V_k^{(\gamma)}.
\]

(61)
Chebyshev Polynomials and Fredholm-Volterra Integral Equation

The formula (46) becomes

$$\mu a_0^{(\gamma)} - \lambda (\ln 2 - d) a_0^{(\gamma)} - \lambda \sum_{k=0}^{\gamma} a_0^{(k)} W_k V_k^{(\gamma)} = f_0^{(\gamma)} + O(h^{p+1}),$$

(62)

where

$$h = \max_{0 \leq k \leq \gamma} h_k \rightarrow 0 \text{ and } h_k = t_{k+1} - t_k.$$  

(63)

The values of the weights $W_k$ are taken in the form

$$W_k = \begin{cases} \frac{h}{2}, & k = \{0, \gamma\} \\ h, & k \neq \{0, \gamma\}, \ k = 0, 1, 2, ..., \gamma. \end{cases}$$

(64)

Also, the error of Eq.(62) and $W_j (j = 1, ..., \gamma)$ depend on the number of the derivatives of $V(|t - \tau|), \tilde{p} \simeq \gamma$ (see (Atkinson 1997; Delves and Mohamed 1985)).

**The total error of this method**

The error involved from Volterra integral term $E_2$, in the light of Eq.(62), can be determined by

$$E_2 = \| \phi_N (x, t) - \phi_N (x, t_i) \| = O(h^{p+1}).$$

(65)

From (44) and (65), we define the total error in the form

$$E = E_1 + E_2 \leq DN^{-r} + O(h^{p+1}).$$

(66)

Now, we can convert the system of VIEs (49), (51) and (54) into a linear algebraic system by the following discussion:

(1) For $n = 0, \ m = 0$, the solution Eq.(62) takes the form

$$a_0^{(\gamma)} = \frac{f_0^{(\gamma)} + \lambda \sum_{k=0}^{\gamma-1} W_k V_k^{(\gamma)} a_0^{(k)}}{\mu - 2\lambda (\ln 2 - d) - \frac{h^p}{2} V^{(\gamma)}},$$

(67)

(2) For $n = 0, \ m \neq 0$ and with the help of Eq.(61), the formula(48) tends to

$$[\mu - 2\lambda \sum_{m=1}^{M} \frac{1 - 2m^2}{m(1 - 4m^2)}] a_0^{(\gamma)} - \lambda \sum_{k=0}^{\gamma} W_k V_k^{(\gamma)} a_0^{(k)} = f_0^{(\gamma)},$$

(68)

therefore, we have

$$a_0^{(\gamma)} = \frac{f_0^{(\gamma)} + \lambda \sum_{k=0}^{\gamma-1} W_k V_k^{(\gamma)} a_0^{(k)}}{\mu - 2\lambda \sum_{m=1}^{M} \frac{1 - 2m^2}{m(1 - 4m^2)} - \frac{h^p}{2} V^{(\gamma)}},$$

(69)

(3) For $n \neq 0, \ m = 0$ and with the aid of (61), Eq.(49) becomes

$$[\mu - \frac{2\lambda}{h} \sum_{l=1}^{N} A_{n,l}] a_n^{(\gamma)} - \lambda \sum_{k=0}^{\gamma} W_k V_k^{(\gamma)} a_n^{(k)} = f_n^{(\gamma)},$$

(70)

hence, we get

$$a_n^{(\gamma)} = \frac{f_n^{(\gamma)} + \lambda \sum_{k=0}^{\gamma-1} W_k V_k^{(\gamma)} a_n^{(k)}}{\mu - \frac{2\lambda}{h} \sum_{l=1}^{N} A_{n,l} - \frac{h^p}{2} V^{(\gamma)}},$$

(71)

For \( n = m \neq 0 \) and using (61), Eq.(51) takes the form
\[
\left[ \mu - \lambda \left( \sum_{l=1}^{N} \frac{A_{n,l}}{4n} + \beta_{n,l}(\ln 2 - d) \right) \right] a_n^{(\gamma)} - \lambda \sum_{k=0}^{\gamma} W_k V_k^{(\gamma)} a_n^{(k)} = f_n^{(\gamma)}, \tag{72}
\]
then, the general solution of (72) is
\[
a_n^{(\gamma)} = \frac{f_n^{(\gamma)} + \lambda \sum_{k=0}^{\gamma-1} W_k V_k^{(\gamma)} a_n^{(k)}}{\mu - \lambda \left( \sum_{l=1}^{N} \frac{A_{n,l}}{4n} + \beta_{n,l}(\ln 2 - d) \right) - \frac{\lambda}{2} V^{(\gamma)}}, \tag{73}
\]
For \( n \neq m \neq 0 \), using (61), Eq.(54) gives
\[
\left[ \mu - \frac{\lambda}{2} \sum_{l=1}^{N} \sum_{m=1}^{M} A_{n,m,l} \right] a_n^{(\gamma)} - \lambda \sum_{k=0}^{\gamma} W_k V_k^{(\gamma)} a_n^{(k)} = f_n^{(\gamma)}, \tag{74}
\]
therefore, we get
\[
a_n^{(\gamma)} = \frac{f_n^{(\gamma)} + \lambda \sum_{k=0}^{\gamma-1} W_k V_k^{(\gamma)} a_n^{(k)}}{\mu - \frac{\lambda}{2} \sum_{l=1}^{N} \sum_{m=1}^{M} A_{n,m,l} - \frac{\lambda}{2} V^{(\gamma)}}. \tag{75}
\]
The formula (75) represents a linear algebraic system.

To prove the existence and uniqueness of this system, we assume

(i) \( \| A_{n,l,m} \| = \sup_{m=1}^{M} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} A_{n,m,l}^{2} \| a_n^{(\gamma)} \| \leq Q^* \)

(ii) \( \| f_n^{(\gamma)} \| = \sup_{\gamma} \left( \sum_{n=1}^{\infty} f_n^{(\gamma)} \right) \leq H^* \)

(iii) \( \max_{\gamma} \sum_{k=0}^{\gamma} V_k W_k \leq L^* \)

where \( Q^* \), \( H^* \) and \( L^* \) are constants.

**Theorem 7.1:**

The linear algebraic system of Eq. (74), when \( N, M \to \infty \) is bounded and has a unique solution under the following condition
\[
|\lambda| < \frac{|\mu|}{\frac{1}{2}Q^* + L^*}. \tag{76}
\]

**Proof:**

The system (74), after applying Cauchy-Minkowski inequality, yields
\[
|\mu| \| a_n^{(\gamma)} \| \leq \| f_n^{(\gamma)} \| + \frac{\lambda}{2} \left( \sum_{l=1}^{N} \sum_{m=1}^{M} A_{n,m,l} \| a_n^{(\gamma)} \| + \| \lambda \| \sum_{k=0}^{\gamma} W_k V_k^{(\gamma)} \| a_n^{(\gamma)} \| . \right) \tag{77}
\]
Using conditions (i)-(iii), we get
\[ \| a_n^{(\gamma)} \| \leq \frac{H^*}{|(|\mu| - |\frac{1}{2}|(Q^* - |\lambda|L^*)|}. \] (78)

To prove \( a_n^{(\gamma)} \) is unique, assume \( \tilde{a}_n^{(\gamma)} \) is another solution of system (74), hence
\[ a_n^{(\gamma)} - \tilde{a}_n^{(\gamma)} = \frac{\lambda}{\mu} \frac{1}{2} \sum_{i=1}^{N} \sum_{m=1}^{M} A_{n,m,i} + \lambda \sum_{k=0}^{\gamma} W_k V_k^{(\gamma)}[a_n^{(\gamma)} - \tilde{a}_n^{(\gamma)}]. \] (79)

Applying Cauchy-Minkowski inequality and conditions (i)-(iii), we get
\[ \| a_n^{(\gamma)} - \tilde{a}_n^{(\gamma)} \| \leq |\frac{\lambda}{\mu}|(\frac{1}{2}(Q^* + L^*)) \| a_n^{(\gamma)} - \tilde{a}_n^{(\gamma)} \|. \] (80)

Using condition (76), we have \( a_n^{(\gamma)} = \tilde{a}_n^{(\gamma)} \). Hence, The solution of the system (74) leads directly to obtain the linear coefficients of Chebyshev polynomial, \( a_n^{(\gamma)} \), therefore, the solution of Fredholm-Volterra integral equation (1) is completely determined by (37).

8 NUMERICAL RESULTS

Using (69) and (75), we can graph \( \phi(x) \) in the region \( x = (-1, 1) \) for different \( T = \{0.7, 0.3\} \), when \( N = 150, M = 10, V = T^2 - \tau^2, f = x^2t^2, d = 0.01, \lambda = .01 \) and \( \mu = 1 \), the graph is shown in Figures 1, 2.

![Graph](image_url)

Figure 1: \([T = 0.7, N = 150, M = 10, V = T^2 - \tau^2, f = x^2t^2, d = 0.01, \lambda = 0.01, \mu = 1]\)

The error term \( E_N \) can be given from the relation
\[ E_N = \| \frac{\phi_{N+1}}{(N+1)!} \| O(N^{-N-1}). \]
Figure 2: $[T = 0.3, N = 150, M = 10, V = T^2 - r^2, f = x^2t^2, d = 0.01, \lambda = 0.01, \mu = 1]$

Figure 3: $[T = 0.7, N = 150, M = 10, V = T^2 - r^2, f = x^2t^2, d = 0.01, \lambda = 0.01, \mu = 1]$
Figure 4: \([T = 0.3, N = 150, M = 10, V = T^2 - \tau^2, f = x^2t^2, d = 0.01, \lambda = 0.01, \mu = 1]\)

If \(N \to \infty\), we have \(E_N \to 0\). This error is calculated where we obtain the result by Figures 3, 4.

Using (69) and (75), If we take \(T = 0.3, M = 10, f = x^2t^2, d = 0.01, \mu = 1\), we can derive
the error function \(E(x)\) at some values of \(x\) as the following

Table 1:

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</table>

9 CONCLUSION

1- The solution \(\phi(x, t)\) is symmetric with respect to \(x\).
2- The error takes maximum value at the ends when \(x = 1\) and \(x = -1\), while it is minimum at
the middle when \(x = 0\).
3- The error function, in figure 4, is less than the corresponding one, in figure 3, by decreasing
the time.
4- For smaller values of \(\lambda\), the solution is stable, while in bigger one, the solution diverges.
5- From the table, we note that the error is stable by increasing \(N\).

REFERENCES


