ITERATIVE METHODS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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ABSTRACT
An iterative method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space has been studied. We prove a strong convergence theorem under mild assumptions.

Keywords: Nonexpansive mapping, Equilibrium problem, Fixed point problem, Hilbert space.

1 INTRODUCTION
Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $h : C \times C \to R$ be an equilibrium bifunction, i.e., $h(u, u) = 0$ for every $u \in C$. Then, one can define the equilibrium problem that is to find an element $u \in C$ such that

$$h(u, v) \geq 0 \quad \text{for all } v \in C. \tag{1}$$

Denote the set of solutions of equilibrium problem (1) by $S(h)$. This problem contains fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems as special cases; please see (Blum and Oettli 1994) for more details. Some methods have been proposed to solve the equilibrium problem, the reader can consult (Combettes and Hirstoaga 2005; Flam and Antipin 1997; Takahashi and Takahashi 2007).

Recently, (Combettes and Hirstoaga 2005) introduced an iterative scheme of finding the best approximation to the initial data when $S(h) \neq \emptyset$ and proved a strong convergence theorem. Motivated by the idea of Combettes and Hirstoaga, very recently (Takahashi and Takahashi 2007) introduced a new iterative method by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Their results extend and improve the corresponding results announced by many authors.

In this paper, motivated and inspired by (Combettes and Hirstoaga 2005) and (Takahashi and Takahashi 2007), we study an iterative method for finding a common element of the set of solutions of problem (1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. We obtain a strong convergence theorem which improve and extend the corresponding results of (Combettes and Hirstoaga 2005; Takahashi and Takahashi 2007).

2 PRELIMINARIES

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C$ be a nonempty closed convex subset of $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|,$$

for all $y \in C$. Such a $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \iff \langle x - x^*, x^* - y \rangle \geq 0 \text{ for all } y \in C.$$

Recall that a mapping $T : C \to H$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. Denote the set of fixed points of $T$ by $F(T)$. It is well known that if $C$ is bounded closed convex and $T : C \to C$ is nonexpansive, then $F(T) \neq \emptyset$. We call a mapping $f : H \to H$ is contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \text{ for all } x, y \in H.$$

For an equilibrium bifunction $h : C \times C \to \mathbb{R}$, we call $h$ satisfying condition (A) if $h$ satisfies the following three conditions:

(a) $h$ is monotone, i.e., $h(x, y) + h(y, x) \leq 0$ for all $x, y \in C$;

(b) for each $x, y, z \in C$, $\lim_{t \to 0} h(tz + (1 - t)x, y) \leq h(x, y)$;

(c) for each $x \in C$, $y \mapsto h(x, y)$ is convex and lower semicontinuous.

If an equilibrium bifunction $h : C \times C \to \mathbb{R}$ satisfies condition (A), then we have the following two important results. You can find the first lemma in (Blum and Oettli 1994) and the second one in (Combettes and Hirstoaga 2005).

**Lemma 2.1.** Let $C$ be a nonempty closed convex subset of $H$ and let $h$ be an equilibrium bifunction of $C \times C$ into $\mathbb{R}$ satisfies condition (A). Let $r > 0$ and $x \in H$. Then, there exists $y \in C$ such that

$$h(y, z) + \frac{1}{r} \langle z - y, y - x \rangle \geq 0 \text{ for all } z \in C.$$

**Lemma 2.2.** Assume that $h$ satisfies the same assumptions as Lemma 2.1. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:

$$S_r(x) = \{ y \in C : h(y, z) + \frac{1}{r} \langle z - y, y - x \rangle \geq 0, \forall z \in C \}$$

for all $y \in H$. Then, the following hold:

(1) \( S_r \) is single-valued;
(2) \( S_r \) is firmly nonexpansive, i.e., for any \( x, y \in H \),
\[
\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle;
\]
(3) \( F(S_r) = S(h) \);
(4) \( S(h) \) is closed and convex.

We also need the following lemmas for proving our main results.

**Lemma 2.3.** (Suzuki 2005) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( X \) and let \( \{\beta_n\} \) be a sequence in \([0, 1]\) with \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \). Suppose \( x_{n+1} = (1-\beta_n)y_n + \beta_n x_n \) for all integers \( n \geq 0 \) and \( \lim \sup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).

**Lemma 2.4.** (Xu 2004) Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that \( a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n \), where \( \{\gamma_n\} \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence such that

(i) \( \sum_{n=1}^{\infty} \gamma_n = \infty \);
(ii) \( \lim \sup_{n \to \infty} \delta_n / \gamma_n \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

### 3 Main Results

Now we state and prove our main results.

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of \( H \). Let \( h : C \times C \to R \) be an equilibrium bifunction satisfying condition (A) and let \( T \) be a nonexpansive mapping of \( C \) into \( H \) such that \( F(T) \cap S(h) \neq \emptyset \). Suppose \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \((0, 1)\) and \( \{r_n\} \subset (0, \infty) \) is a real sequence. Suppose the following conditions are satisfied:

(C1) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \);
(C2) \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \);
(C3) \( \lim \inf_{n \to \infty} r_n > 0 \) and \( \lim \sup_{n \to \infty} (r_{n+1} - r_n) = 0 \).

Let \( f \) be a contraction of \( H \) into itself and given \( x_0 \in H \) arbitrarily. Let \( \{x_n\} \) be sequence generated by

\[
\begin{cases}
  h(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \ \forall y \in C \\
  z_n = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \\
  x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n.
\end{cases}
\]

Then the sequence \( \{x_n\} \) converges strongly to \( x^* \in F(T) \cap S(h), \) where \( x^* = P_{F(T) \cap S(h)}f(x^*) \).

Proof. Let $Q = P_{F(T) \cap S(h)}$. Note that $f$ is a contraction mapping with coefficient $\alpha \in (0, 1)$. Then $\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in H$. Therefore, $Qf$ is a contraction of $H$ into itself, which implies that there exists a unique element $x^* \in H$ such that $x^* = Qf(x^*)$. At the same time, we note that $x^* \in C$.

Let $p \in F(T) \cap S(h)$. From the definition of $S_\tau$, we note that $u_n = S_\tau x_n$. It follows that

$$\|u_n - p\| = \|S_\tau x_n - S_\tau p\| \leq \|x_n - p\|.$$

Next we prove that $\{x_n\}$ and $\{u_n\}$ are bounded. Indeed, from (2), we obtain

$$\|x_{n+1} - p\| = \|\beta_n(x_n - p) + (1 - \beta_n)(z_n - p)\| \leq \beta_n\|x_n - p\| + (1 - \beta_n)\|z_n - p\| = \beta_n\|x_n - p\| + (1 - \beta_n)(\|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(Tu_n - p)\| \leq \beta_n\|x_n - p\| + (1 - \beta_n)\|\alpha_n f(x_n) - p\| + (1 - \alpha_n)\|Tu_n - p\| \leq \beta_n\|x_n - p\| + (1 - \beta_n)\|\alpha_n f(x_n) - f(p)\| + \|f(p) - p\| + (1 - \beta_n)(1 - \alpha_n)\|u_n - p\| \leq \beta_n\|x_n - p\| + (1 - \beta_n)\|\alpha_n f(x_n) - p\| + \|f(p) - p\| + (1 - \beta_n)(1 - \alpha_n)\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1 - \alpha}\|f(p) - p\|\}.$$

Therefore $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{Tx_n\}$ and $\{f(x_n)\}$ are all bounded. From (2), we note that

$$z_{n+1} - z_n = \alpha_n + 1 f(x_{n+1}) - \alpha_n f(x_n) + (1 - \alpha_n) Tu_{n+1} - (1 - \alpha_n) Tu_n = \alpha_n + 1 f(x_{n+1}) - \alpha_n f(x_n) + (1 - \alpha_n) (Tu_{n+1} - Tu_n) + (\alpha_n - \alpha_n + 1) Tu_n.$$

It follows that

$$\|z_{n+1} - z_n\| \leq \alpha_n + 1 \|f(x_{n+1})\| + \alpha_n \|f(x_n)\| + (1 - \alpha_n) \|u_{n+1} - u_n\| + |\alpha_n - \alpha_n + 1| \|Tu_n\|.$$

On the other hand, from $u_n = S_\tau x_n$ and $u_{n+1} = S_\tau x_{n+1}$, we have

$$h(u_n, x) + \frac{1}{r_n} (x - u_n, u_n - x) \geq 0 \text{ for all } x \in C$$

and

$$h(u_{n+1}, x) + \frac{1}{r_{n+1}} (x - u_{n+1}, u_{n+1} - x_{n+1}) \geq 0 \text{ for all } x \in C.$$

Putting $x = u_{n+1}$ in (4) and $x = u_n$ in (5), we have

$$h(u_n, u_{n+1}) + \frac{1}{r_n} (u_{n+1} - u_n, u_n - x) \geq 0,$$

and

$$h(u_{n+1}, u_n) + \frac{1}{r_{n+1}} (u_n - u_{n+1}, u_{n+1} - x_{n+1}) \geq 0.$$

From the monotonicity of $h$, we have

$$h(u_n, u_{n+1}) + h(u_{n+1}, u_n) \leq 0.$$  

So, from (6) and (7), we can conclude that

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \rangle \geq 0.$$  

Since $\liminf_{n \to \infty} r_n > 0$, without loss of generality, we may assume that there exists a real number $b$ such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\|u_{n+1} - u_n\|^2 \leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle$$

$$\leq \|u_{n+1} - u_n\| \{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{M}{b}|r_{n+1} - r_n|,$$

where $M$ is a constant such that $\|u_{n+1} - x_{n+1}\| \leq M$ for all $n \geq 0$. Substituting (8) into (3), we have

$$\|z_{n+1} - z_n\| \leq \alpha_{n+1}\|f(x_{n+1})\| + \alpha_n\|f(x_n)\| + (1 - \alpha_{n+1})\{\|x_{n+1} - x_n\|$$

$$+ \frac{M}{b}|r_{n+1} - r_n|\} + |\alpha_n - \alpha_{n+1}|\|Tu_n\|$$

this together with $\alpha_n \to 0$ and $r_{n+1} - r_n \to 0$ imply that

$$\limsup_{n \to \infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$  

Hence by Lemma 2.3, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$  

From (8) and $\lim_{n \to \infty}(r_{n+1} - r_n) = 0$, we have

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0.$$  

We note that

$$\|x_n - Tu_n\| \leq \|x_n - z_n\| + \|z_n - Tu_n\|$$

$$\leq \|x_n - z_n\| + \alpha_n\|f(x_n) - Tu_n\|$$

$$\to 0.$$  

For $p \in F(T) \cap S(h)$, note that $S_r$ is firmly nonexpansive, then we have

$$\|u_n - p\|^2 = \|S_{r_n}x_n - S_{r_n}p\|^2$$

$$\leq \langle S_{r_n}x_n - S_{r_n}p, x_n - p \rangle$$

$$= \langle u_n - p, x_n - p \rangle$$

$$= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2)$$

and hence
\[ \|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \]

Therefore, we have
\[
\|x_{n+1} - p\|^2 = \|\beta_n x_n + (1 - \beta_n) z_n - p\|^2 \\
\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\
= \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(Tu_n - p)\|^2 \\
\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|\alpha_n f(x_n) - p\|^2 + (1 - \alpha_n)\|Tu_n - p\|^2 \\
\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)(1 - \alpha_n)(\|x_n - p\|^2 - \|x_n - u_n\|^2).
\]

Then we have
\[
(1 - \beta_n)(1 - \alpha_n)\|x_n - u_n\|^2 \leq (1 - \beta_n)\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
\leq \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \\
\times (\|x_n - p\| - \|x_{n+1} - p\|) \\
\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\]

It is easily seen that \( \lim \inf_{n \to \infty} (1 - \beta_n)(1 - \alpha_n) > 0 \). So, we have that \( \lim_{n \to \infty} \|x_n - u_n\| = 0 \). From
\[
\|Tu_n - u_n\| \leq \|Tu_n - x_n\| + \|x_n - u_n\|.
\]

We also have \( \|Tu_n - u_n\| \to 0 \). Next, we show that
\[
\lim_{n \to \infty} \sup \langle f(x^*) - x^*, x_n - x^* \rangle \leq 0,
\]
where \( x^* = P_{F(T) \cap S(h)} f(x^*) \). First we can choose a subsequence \( \{u_{n_j}\} \) of \( \{u_n\} \) such that
\[
\lim_{j \to \infty} \langle f(x^*) - x^*, u_{n_j} - x^* \rangle = \lim_{n \to \infty} \sup \langle f(x^*) - x^*, u_n - x^* \rangle.
\]

Since \( \{u_{n_j}\} \) is bounded, there exists a subsequence \( \{u_{n_{j_i}}\} \) of \( \{u_{n_j}\} \) which converges weakly to \( w \). Without loss of generality, we can assume that \( u_{n_j} \to w \) weakly. From \( \|Tu_n - u_n\| \to 0 \), we obtain \( Tu_{n_j} \to w \) weakly. Now we show \( w \in S(h) \). By \( u_n = S_{r_n} x_n \), we have
\[
h(u_n, x) + \frac{1}{r_n} (x - u_n, u_n - x_n) \geq 0, \quad \forall x \in C.
\]

From the monotonicity of \( h \), we have
\[
\frac{1}{r_n} (x - u_n, u_n - x_n) \geq -h(u_n, x) \geq h(x, u_n),
\]
and hence
\[
\langle x - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \rangle \geq h(x, u_{n_j}).
\]

Since \( \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \to 0 \) and \( u_{n_j} \to w \) weakly, from the lower semi-continuity of \( h(x, y) \) on the second variable \( y \), we have
\[
h(x, w) \leq 0
\]

for all $x \in C$. For $t$ with $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)w$. Since $x \in C$ and $w \in C$, we have $x_t \in C$ and hence $h(x_t, w) \leq 0$. So, from the convexity of equilibrium bifunction $h(x, y)$ on the second variable $y$, we have

$$0 = h(x_t, x_t)$$
$$\leq th(x_t, x) + (1-t)h(x_t, w)$$
$$\leq th(x_t, x).$$

and hence $h(x_t, x) \geq 0$. Then, we have

$$h(w, x) \geq 0$$

for all $x \in C$ and hence $w \in S(h)$.

We shall show $w \in F(T)$. Assume $w \notin F(T)$. Since $u_{n_j} \rightharpoonup w$ weakly and $w \neq Tw$, from Opial’s condition, we have

$$\liminf_{j \to \infty} \|u_{n_j} - w\| < \liminf_{j \to \infty} \|u_{n_j} - Tw\|$$
$$\leq \liminf_{j \to \infty} (\|u_{n_j} - Tu_{n_j}\| + \|Tu_{n_j} - Tw\|)$$
$$\leq \liminf_{j \to \infty} \|u_{n_j} - w\|. $$

This is a contradiction. So, we get $w \in F(T)$. Therefore $w \in F(T) \cap S(h)$. Since $x^* = P_{F(T) \cap S(h)} f(x^*)$, we have

$$\limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle = \lim_{j \to \infty} \langle f(x^*) - x^*, x_{n_j} - x^* \rangle$$
$$= \lim_{j \to \infty} \langle f(x^*) - x^*, u_{n_j} - x^* \rangle$$
$$= \langle f(x^*) - x^*, w - x^* \rangle \leq 0,$$

which implies (noting that $\|x_n - z_n\| \to 0$)

$$\limsup_{n \to \infty} \langle f(x^*) - x^*, z_n - x^* \rangle \leq 0. \tag{9}$$

Finally, we prove that $\{x_n\}$ converges strongly to $x^*$. From (2), we have

$$\|z_n - x^*\|^2 = \|\alpha_n(f(x_n) - x^*) + (1 - \alpha_n)(Tu_n - x^*)\|^2$$
$$\leq (1 - \alpha_n)^2 \|Tu_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - x^*, z_n - x^* \rangle$$
$$\leq (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2\alpha_n \langle (f(x_n) - f(x^*)), z_n - x^* \rangle + \langle f(x^*) - x^*, z_n - x^* \rangle$$
$$\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \|x_n - x^*\| \|z_n - x^*\|$$
$$+ 2\alpha_n \langle f(x^*) - x^*, z_n - x^* \rangle$$
$$\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + \alpha_n \|x_n - x^*\|^2 + \alpha_n \|z_n - x^*\|^2$$
$$+ 2\alpha_n \langle f(x^*) - x^*, z_n - x^* \rangle,$$

that is

$$\|z_n - x^*\|^2 \leq \frac{(1 - \alpha_n)^2 + \alpha_n \|x_n - x^*\|^2}{1 - \alpha_n} + \frac{2\alpha_n}{1 - \alpha_n} \langle f(x^*) - x^*, z_n - x^* \rangle. \tag{10}$$

From (2) and (10), we have
\[
\|x_{n+1} - x^*\|^2 \leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|z_n - x^*\|^2 \\
\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left(1 - \alpha_n\right)\frac{2\alpha_n}{1 - \alpha_n} f(x^*) - x^*, z_n - x^*) \\
(1 - \beta_n) \left(1 - \alpha_n\right)\frac{2\alpha_n}{1 - \alpha_n} f(x^*) - x^*, z_n - x^*) \\
\leq (1 - \beta_n) \left(1 - \alpha_n\right)\frac{2\alpha_n}{1 - \alpha_n} f(x^*) - x^*, z_n - x^*) \\
\leq (1 - \delta_n) \|x_n - x^*\|^2 + \delta_n \sigma_n,
\]
where \(\delta_n = \frac{2(1 - \alpha)(1 - \beta_n)\alpha_n}{1 - \alpha_n}\) and \(\sigma_n = \frac{\|x_n - x^*\|^2\alpha_n}{2(1 - \alpha)} + \frac{1}{1 - \alpha} f(x^*) - x^*, x_{n+1} - x^*\). It is easily seen that \(\sum_{n=0}^{\infty} \delta_n = \infty\) and \(\limsup_{n \to \infty} \sigma_n \leq 0\). Now applying Lemma 2.4 and (9) to (11) concludes that \(x_n \to x^*(n \to \infty)\). This completes the proof.

As direct consequences of Theorem 3.1, we can obtain easily two corollaries as follows.

**Corollary 3.2.** Let \(C\) be a nonempty closed convex subset of \(H\). Let \(h : C \times C \to R\) be an equilibrium bifunction satisfying condition (A) and let \(T\) be a nonexpansive mapping of \(C\) into \(H\) such that \(F(T) \cap S(h) \neq \emptyset\). Suppose \(\{\alpha_n\}\) and \(\{\beta_n\}\) are two sequences in \((0, 1)\) and \(\{r_n\} \subset (0, \infty)\) is a real sequence. Suppose the following conditions are satisfied:

1. \((C1)\) \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=0}^{\infty} \alpha_n = \infty\);
2. \((C2)\) \(0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1\);
3. \((C3)\) \(\liminf_{n \to \infty} r_n > 0\) and \(\lim_{n \to \infty} (r_{n+1} - r_n) = 0\).

For given \(x_0 \in H\) arbitrarily and fixed \(u \in H\), let \(\{x_n\}\) be sequence generated by

\[
\begin{align*}
&h(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \forall y \in C \\
z_n = \alpha_n u + (1 - \alpha_n) Tu_n, \\
x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n.
\end{align*}
\]

Then the sequence \(\{x_n\}\) converges strongly to \(x^* \in F(T) \cap S(h)\), where \(x^* = P_{F(T) \cap S(h)} u\).
4 APPLICATIONS

Using Theorem 3.1, we prove the following theorems in Hilbert space.

**Theorem 4.1.** Let $C$ be a nonempty closed convex subset of $H$. Let $h : C \times C \to R$ be an equilibrium bifunction satisfying condition (A) such that $S(h) \neq \emptyset$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ and $\{r_n\} \subset (0, \infty)$ is a real sequence. Suppose the following conditions are satisfied:

(C1) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1$;

(C3) $\lim \inf_{n \to \infty} r_n > 0$ and $\lim_{n \to \infty} (r_{n+1} - r_n) = 0$.

Let $f$ be a contraction of $H$ into itself and given $x_0 \in H$ arbitrarily. Let $\{x_n\}$ be sequence generated by

$$
\begin{cases}
  h(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C \\
  z_n = \alpha_n f(x_n) + (1 - \alpha_n) u_n, \\
  x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n.
\end{cases}
$$

Then the sequence $\{x_n\}$ converges strongly to $x^* \in S(h)$, where $x^* = P_{S(h)} f(x^*)$.

**Proof.** Put $Tx = x$ for all $x \in C$ and $r_n = 1$ in Theorem 3.1. Then, from Theorem 3.1 the sequence $\{x_n\}$ converges strongly to $x^* = P_{S(h)} f(x^*)$.

**Theorem 4.2.** Let $C$ be a nonempty closed convex subset of $H$. Let $T$ be a nonexpansive mapping of $C$ into $H$ such that $F(T) \neq \emptyset$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Suppose the following conditions are satisfied:

(C1) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1$.

Let $f$ be a contraction of $H$ into itself and given $x_0 \in H$ arbitrarily. Let $\{x_n\}$ be sequence generated by

$$
\begin{cases}
  z_n = \alpha_n f(x_n) + (1 - \alpha_n) T_P C x_n, \\
  x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n.
\end{cases}
$$

Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$, where $x^* = P_{F(T)} f(x^*)$.

**Proof.** Take $h(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \geq 1$. Then, we have $u_n = P_C x_n = x_n$. So, from Theorem 3.1 the sequence $\{x_n\}$ generated in Theorem 4.1 converges strongly to $P_{F(T)} f(x^*)$.

**Corollary 4.3.** Let $C$ be a nonempty closed convex subset of $H$. Let $T$ be a nonexpansive mapping of $C$ into $H$ such that $F(T) \neq \emptyset$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Suppose the following conditions are satisfied:

\( (C1) \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty; \)

\( (C2) \quad 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1. \)

Given \( x_0 \in H \) arbitrarily and \( u \in H \), let \( \{x_n\} \) be sequence generated by

\[
\begin{align*}
  z_n &= \alpha_n u + (1 - \alpha_n) T_P C x_n, \\
  x_{n+1} &= \beta_n x_n + (1 - \beta_n) z_n.
\end{align*}
\]

Then the sequence \( \{x_n\} \) converges strongly to \( x^* \in F(T) \), where \( x^* = P_{F(T)} u \).

**Corollary 4.4.** Let \( C \) be a nonempty closed convex subset of \( H \). Let \( T \) be a nonexpansive mapping of \( C \) into \( C \) such that \( F(T) \neq \emptyset \). Suppose \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \( (0, 1) \).

Suppose the following conditions are satisfied:

\( (C1) \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty; \)

\( (C2) \quad 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1. \)

Given \( x_0 \in C \) arbitrarily and \( u \in C \), let \( \{x_n\} \) be sequence generated by

\[
\begin{align*}
  z_n &= \alpha_n u + (1 - \alpha_n) T x_n, \\
  x_{n+1} &= \beta_n x_n + (1 - \beta_n) z_n,
\end{align*}
\]

Then the sequence \( \{x_n\} \) converges strongly to \( x^* \in F(T) \), where \( x^* = P_{F(T)} u \).

**Remark 4.5.** We note that our result has strong convergence for nonexpansive mappings with mild assumptions imposed on algorithm parameters. Corollary 4.4 improves and extends the corresponding result in (Xu 2004).

**Example 4.6.** Let \( T : C \to C \) be a nonexpansive mapping. Take \( h(x, y) = 0 \) for all \( x, y \in C \) and \( r_n = 1 \) for all \( n \geq 1 \). Hence, we can take \( \alpha_n = \frac{1}{n} \) and \( \beta_n = \frac{1}{2} \) for all \( n \geq 1 \). By using the Corollary 4.4, the iterative sequence \( \{x_n\} \) defined by

\[
\begin{align*}
  z_n &= \frac{1}{n} u + (1 - \frac{1}{n}) T x_n, \\
  x_{n+1} &= \frac{1}{2} x_n + \frac{1}{2} z_n,
\end{align*}
\]

converges strongly to some fixed point of \( T \).

In particular, let \( H = R^2 \) and define \( T : R^2 \to R^2 \) by

\[
T(r e^{i\theta}) = r e^{i(\theta + \frac{\pi}{2})},
\]

and take \( u = e^{i\pi} \) is a fix element in \( C \). It is obvious that \( T \) is a nonexpansive mapping with a unique fixed point \( x^* = 0 \). In this case, (12) becomes that

\[
\begin{align*}
  z_n &= \frac{1}{n} e^{i\pi} + (1 - \frac{1}{n}) r_n e^{i(\theta_n + \frac{\pi}{2})}, \\
  x_{n+1} &= \frac{1}{2} r_n e^{i\theta_n} + \frac{1}{2} z_n.
\end{align*}
\]

We can rewrite the above equation as

\[ x_{n+1} = \frac{1}{2} r_ne^{i\theta_n} + \frac{1}{2n} e^{i\pi} + \frac{1}{2} (1 - \frac{1}{n}) r_ne^{i(\theta_n + \frac{\pi}{2})}. \]

It is clear that the complex number sequence \( \{x_n\} \) converges strongly to a fixed point \( x^* = 0 \).

REFERENCES


